Adaptive Truncated Sequential Tests and the Bonferroni Procedure

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A Simple Method to Construct Sequential Tests

- $X_1, \ldots, X_n \ldots$ i.i.d. random variables
- $\mathbf{X}_k = (X_1, \ldots, X_k)$ first k observations
- $f : \mathbb{R}^n \to \{0\} \cup [1, \infty)$ with $E_{H_0}\{f(\mathbf{X}_n)\} = \alpha$

A truncated sequential test

Reject H_0 after the k-th observation, if

$$E_{H_0}\{f(\mathbf{X}_n) | \mathbf{X}_k\} \geq 1$$

1. Type I error rate $\leq \alpha$
2. Under appropriate conditions:
   Type I error rate $\to \alpha$ as $n \to \infty$

Some comments

- $E\{f(\mathbf{X}_n)|\mathbf{X}_n) = f(\mathbf{X}_n)$.
- In the final analysis the test rejects whenever

$$f(\mathbf{X}_n) \geq 1.$$

- Let $\varphi \in \{0, 1\}$ be the decision function of any test. With $f(\mathbf{X}_n) = \varphi$ the proposed procedure (typically) gives the fixed sample test.

Example

Testing Scenario

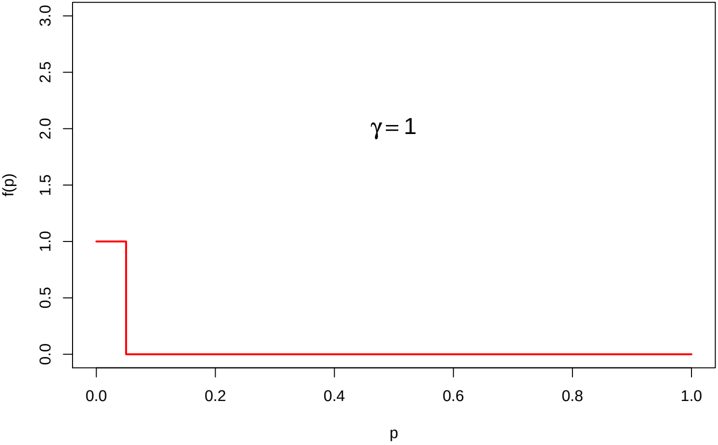
- Test of

$$H_0 : \mu = 0 \text{ against } H_1 : \mu > 0$$

  for the mean of i.i.d. $N(\mu, 1)$ distributed observations.
- $n \ldots$ maximal sample size
- $p = 1 - \Phi(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_n) \ldots$ p-value of fixed sample z-test

A Family of Sequential Tests

$$f(p) = \begin{cases} \gamma & \text{if } p \le \alpha/\gamma \\ 0 & \text{otherwise} \end{cases} \qquad (\gamma \ge 1)$$



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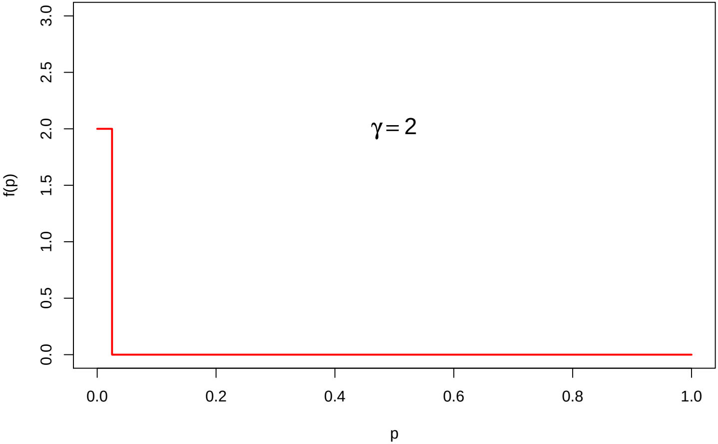

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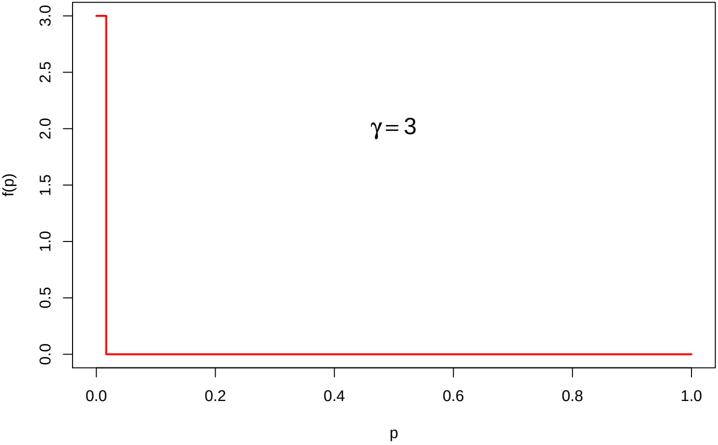


Level and Conditional Expectation

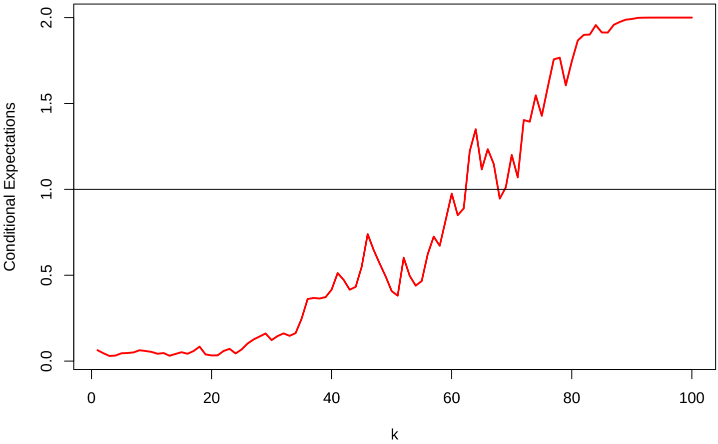
$$f(p) = \begin{cases} \gamma & \text{if } p \leq \alpha/\gamma \\ 0 & \text{otherwise} \end{cases}$$

- $E_{H_0}\{f(p)\} = \gamma \frac{\alpha}{\gamma} = \alpha$
- $E_{H_0}\{f(p)|\mathbf{X}_k\} = \gamma \left[1 - \Phi\left(\frac{z_{1-\alpha/\gamma} - \sqrt{\frac{1}{n}} \sum_{i=1}^{t} X_k}{\sqrt{1-\frac{k}{n}}}\right)\right]$

Corresponds to stochastic curtailment stopping rule (LAN, SIMON, HALPERIN, 1982).

An example path

$$f(p) = \begin{cases} 2 & \text{if } p \leq \alpha/2 \\ 0 & \text{otherwise} \end{cases}$$



Sequential Boundaries for the partial sums $\sum_{i=1}^{k} X_i$

$$f(p) = \begin{cases} \gamma & \text{if } p \le \alpha/\gamma \\ 0 & \text{otherwise} \end{cases}$$

- $E_{H_0}(f(p)|\mathbf{X}_k) = 1 \Leftrightarrow$

$$\sum_{i=1}^{k} X_i = z_{1-\alpha/\gamma}\sqrt{n} - z_{1-1/\gamma}\sqrt{n-k} = a(k)$$

- Special cases:
  - $\gamma = 1$: fixed sample case
  - $\gamma = 2$: $a(k) = z_{1-\alpha/2}$

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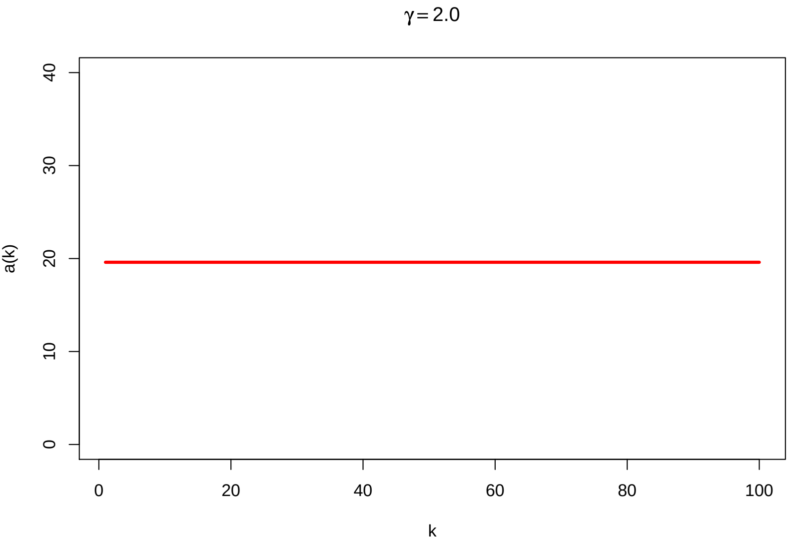

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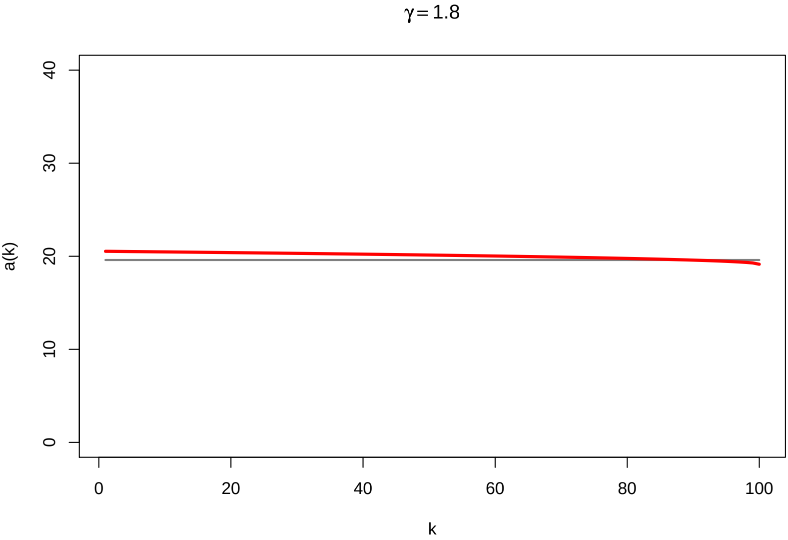

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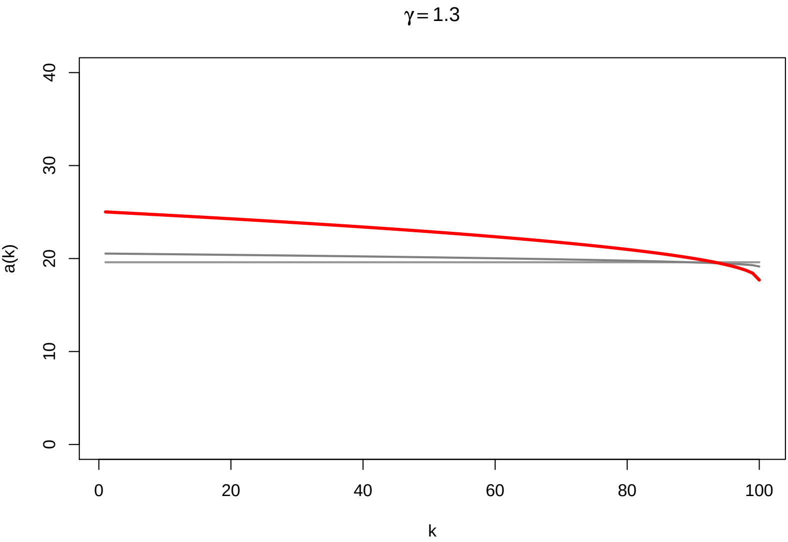

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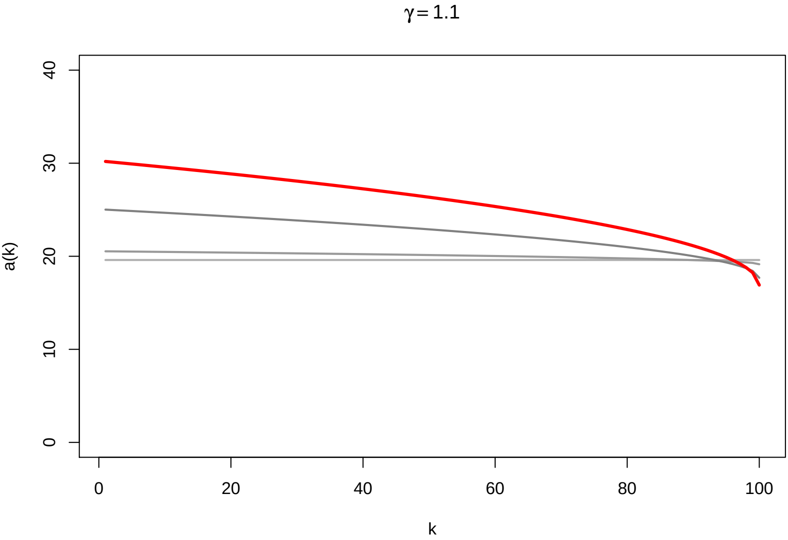

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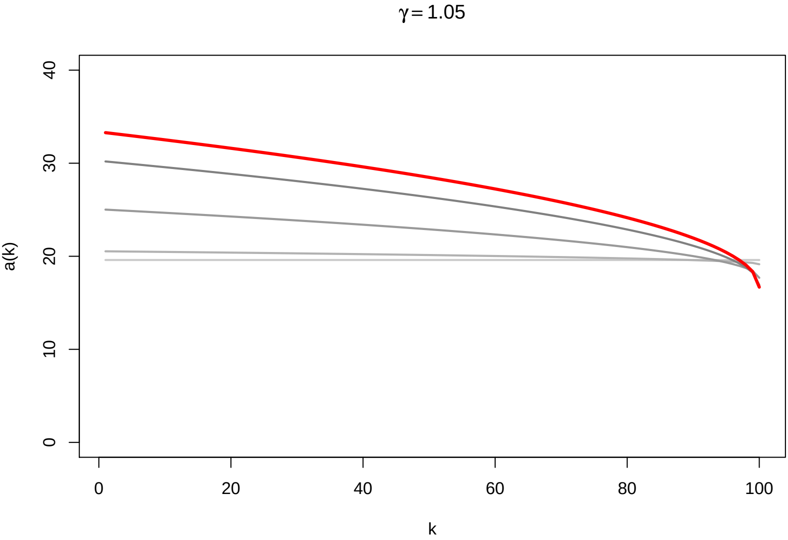

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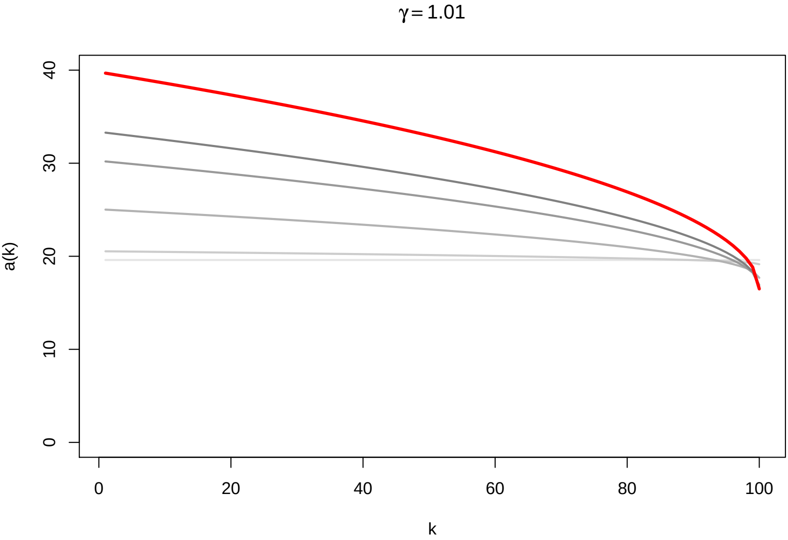

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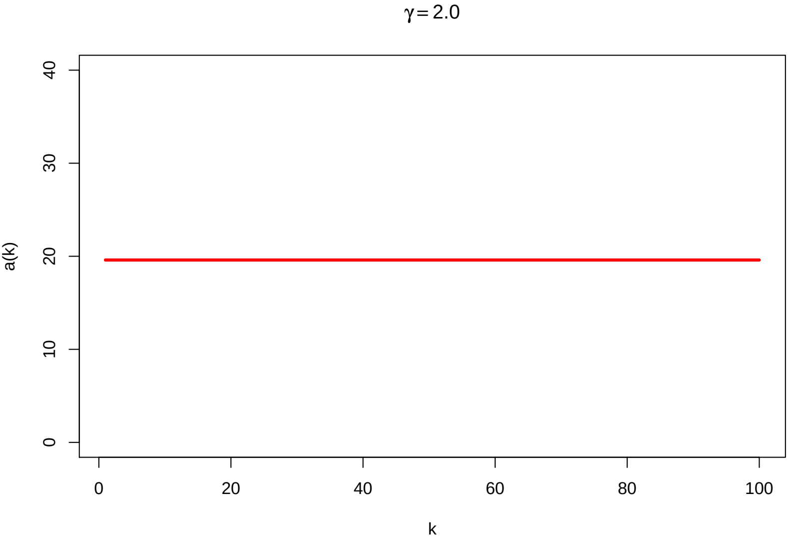

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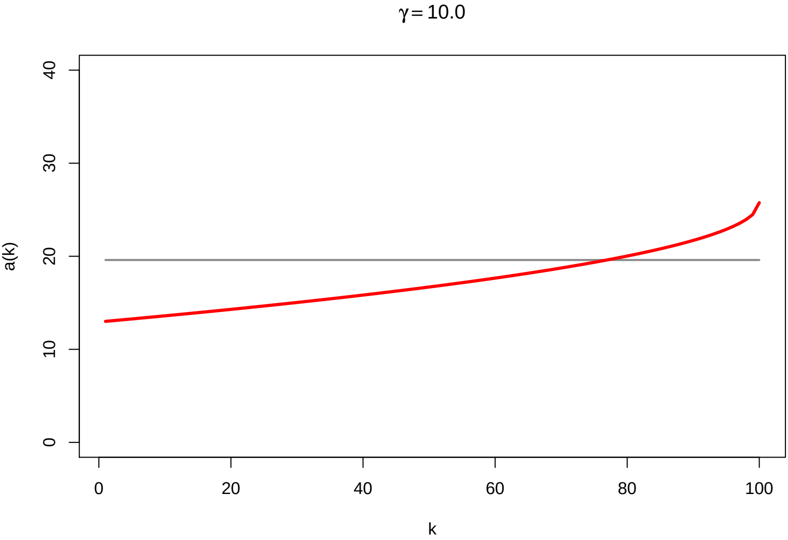

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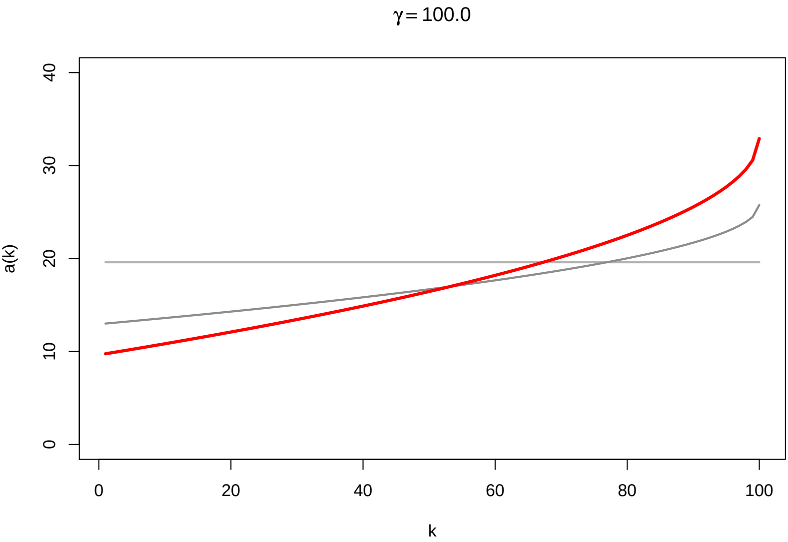

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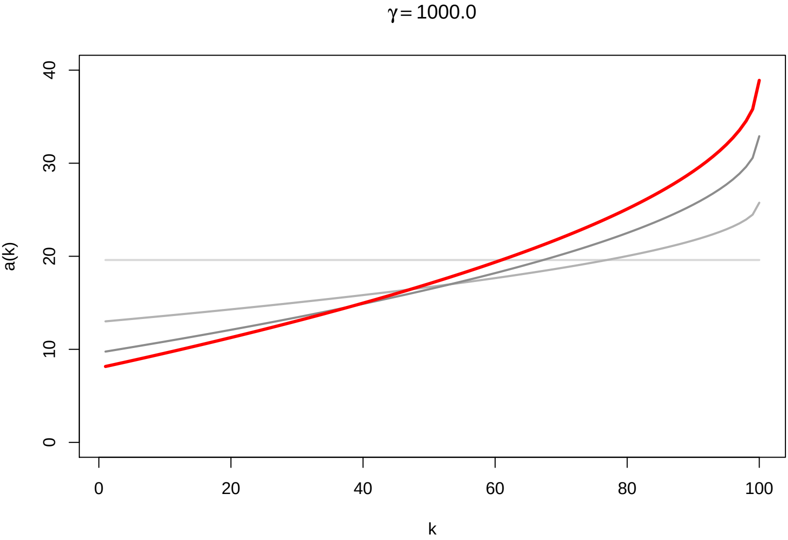

The sequential procedure controls the Type I error rate

Let $\psi_k := E_{H_0}\{f(\mathbf{X}_n)|\mathbf{X}_k\}$ and define the stopping time

$$T := \begin{cases} n & \text{if all } \psi_k < 1 \\ \min(k \ : \ \psi_k \geq 1) & \text{otherwise} \end{cases}$$

The sequential test is given by

$$\varphi = \min(\psi_T, 1)$$

Proof:

ψ_k is a martingale. Optional stopping theorem:
$P(\varphi = 1) = E\{\min(\psi_T, 1)\} \leq E\{\psi_T\} = \psi_0 = E\{f(\mathbf{X}_n)\} = \alpha$

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Asymptotics

Proposition

Assume f is a bounded, monotonic function of an *asymptotically linear* test statistics.

Then asymptotically (as n→ ∞)

1. the sequential test has level α.
2. Assume $P_{H_0}\{f(\mathbf{X}_n) = 1\} = 0$. Then

$$P\{T < n | H_0 \text{ is rejected }\} = 1.$$

Applications

Sign Test, Binomial Test, z-Tests, t-Tests, Likelihood Ratio Tests

Sequential Tests with Futility Stopping

- $f : \mathbb{R}^n \to (-\infty, 0\} \cup [1, \infty)$ with $E_{H_0}\{f(\mathbf{X}_n)\} = \alpha$
- Reject H_0 after the k-th observation, if

$$E_{H_0}\{f(\mathbf{X}_n) | \mathbf{X}_k\} \geq 1$$

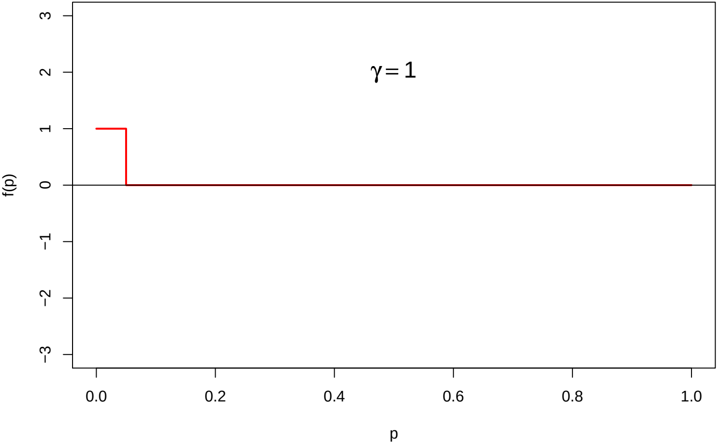
- Accept H_0 after the k-th observation, if

$$E_{H_0}\{f(\mathbf{X}_n) | \mathbf{X}_k\} \leq 0$$

- For asymptotically linear test statistics:
  Type I error rate $\to \alpha$ for $n \to \infty$

A Family of Tests with Futility Stopping

$$f(p) = \begin{cases} \gamma & \text{if } p \le \alpha \\ -\alpha\frac{\gamma-1}{1-\alpha} & \text{otherwise} \end{cases}, \quad E_{H_0}\{f(p)\} = \alpha$$



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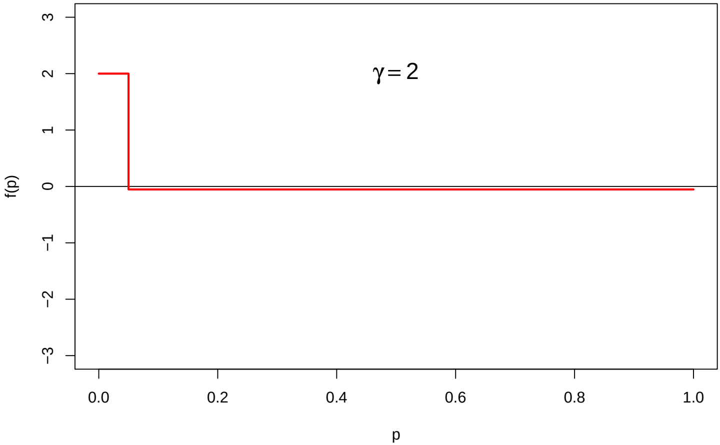

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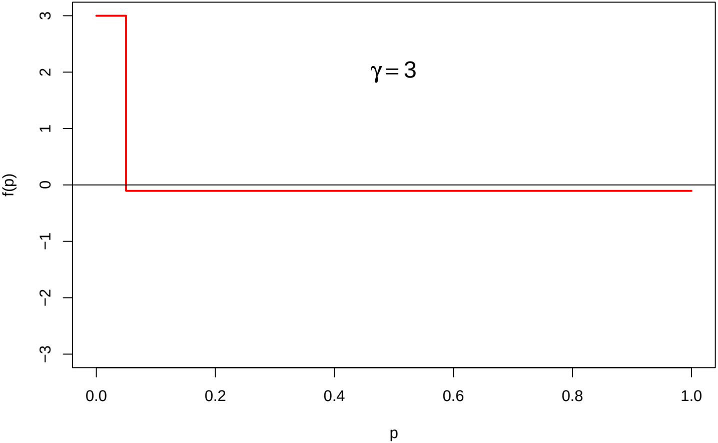

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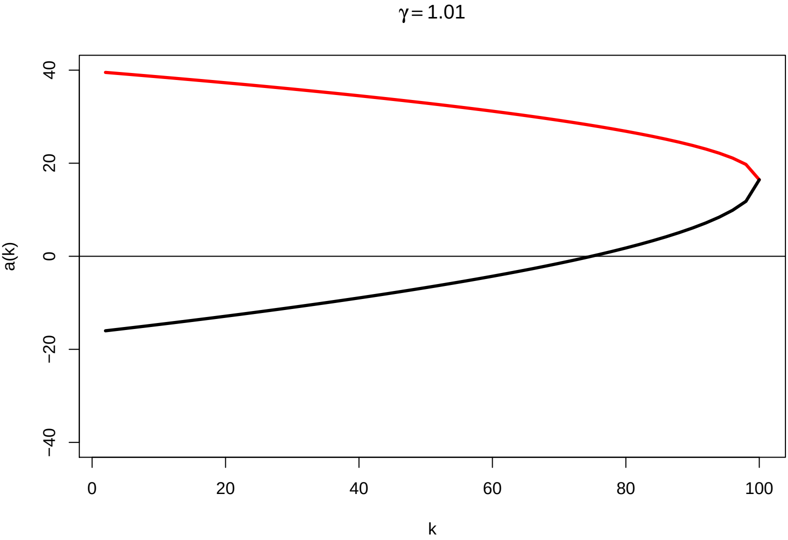

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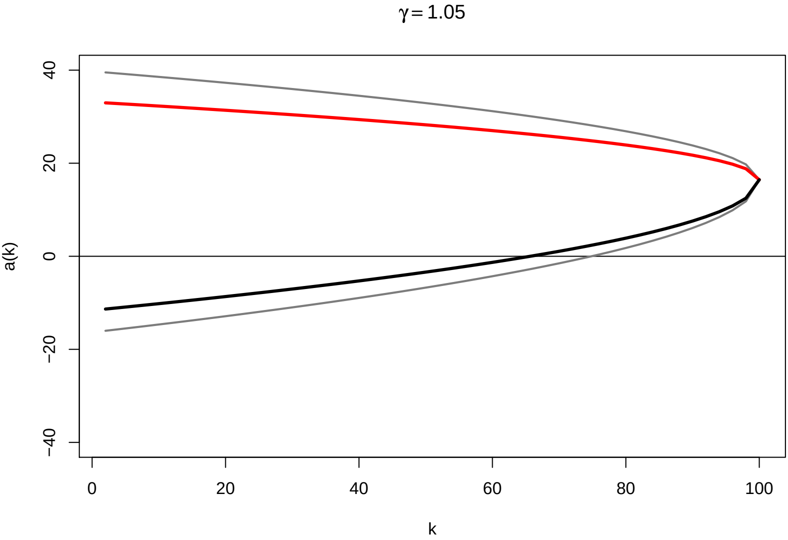

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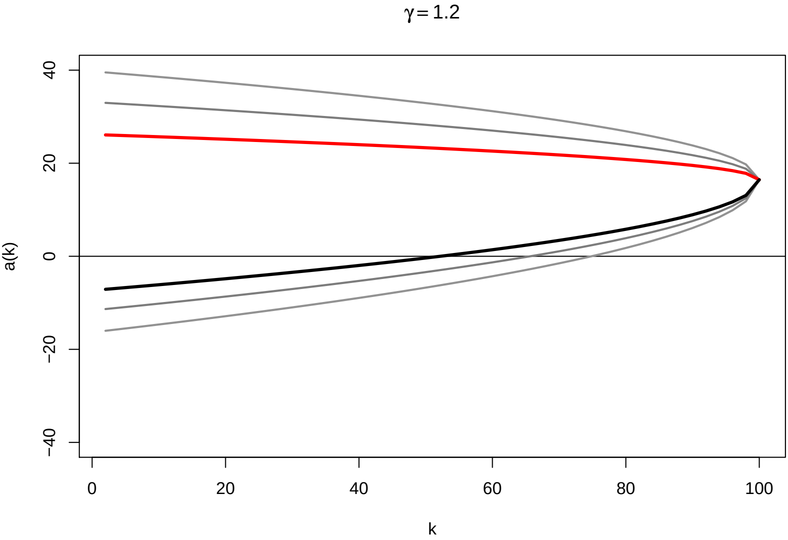

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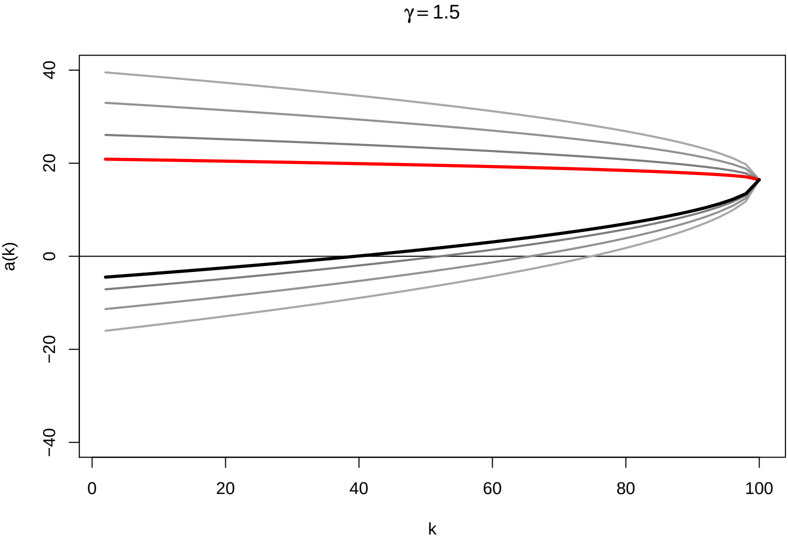

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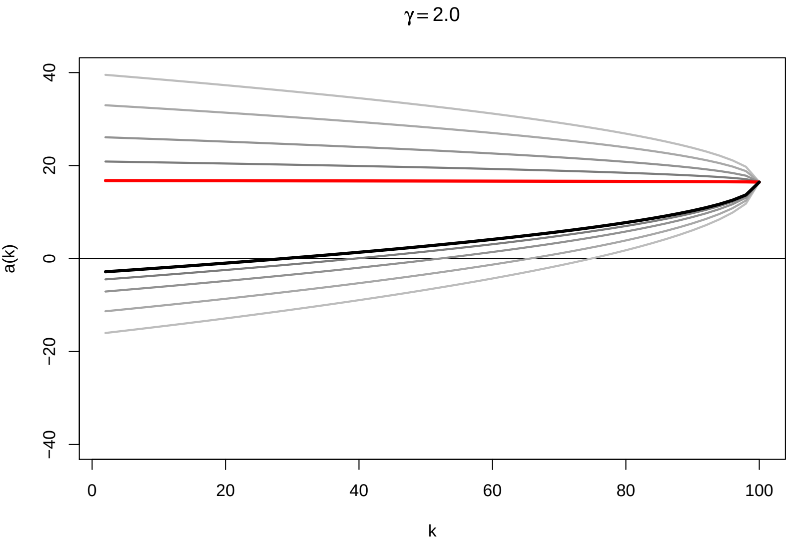

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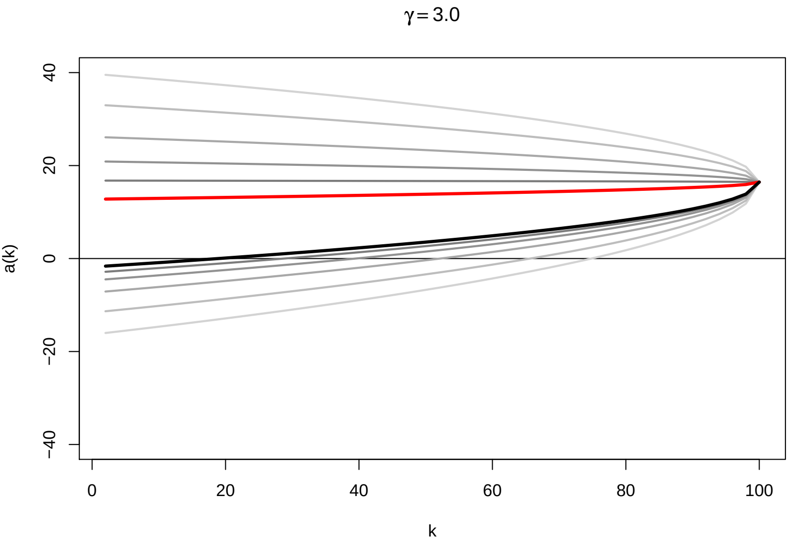

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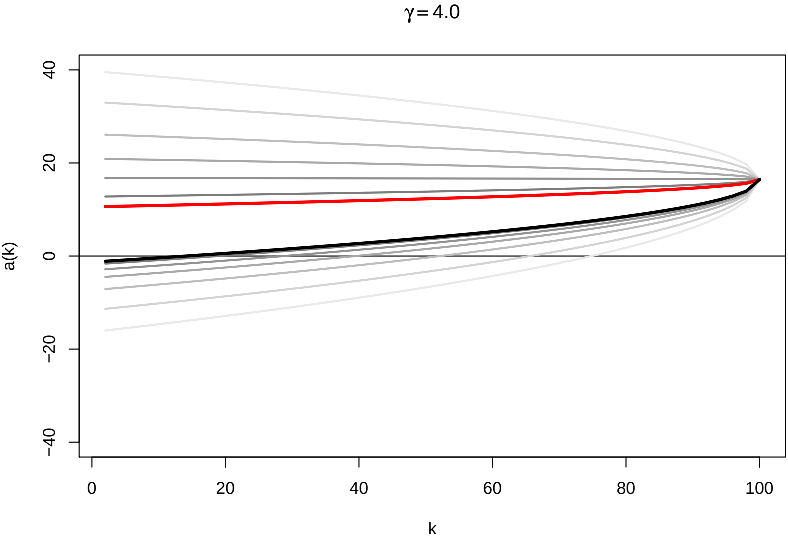


Uniform Improvement of the Bonferroni Test

- $H_A, H_B \ldots$ considered null hypotheses
- $\mathbf{X}_n^A, \mathbf{X}_n^B \ldots$ data vectors for H_A, H_B.
- $p_A, p_B \ldots$ univariate p-values for H_A, H_B
- The Bonferroni test for $H = H_A \cap H_B$ is given by

$$\varphi^B = \min\{\mathbf{1}_{\{p_A \leq \alpha/2\}}, \mathbf{1}_{\{p_B \leq \alpha/2\}}\}.$$

- Let $f(p_A, p_B) = \mathbf{1}_{\{p_A \leq \alpha/2\}} + \mathbf{1}_{\{p_B \leq \alpha/2\}}$.
- $E_H\{f(p_A, p_B)\} = \alpha$, whatever the dependency structure.
- $\varphi^B \leq f(p_A, p_B)$

Uniform Improvement of the Bonferroni Test

Sequential improvement of the Bonferroni Test

Reject $H = H_A \cap H_B$ after the k-th observation, if

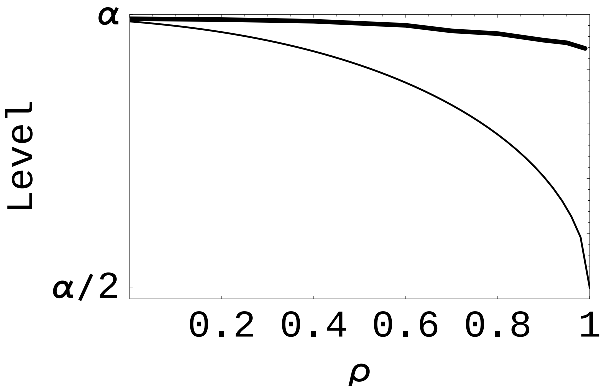
$$E_H[\mathbf{1}_{\{p_A \leq \alpha/2\}} | \mathbf{X}_k^A] + E_H[\mathbf{1}_{\{p_B \leq \alpha/2\}} | \mathbf{X}_k^B] \geq 1$$

- Rejects whenever the classical Bonferroni test rejects
- Compared to the fixed sample Bonferroni test:
  - higher power
  - lower expected sample size
  - asymptotically exhausts the level for all dependence structures

Simulation Study

- Tests for the means of normal data
- Bivariate normal data with correlation ρ

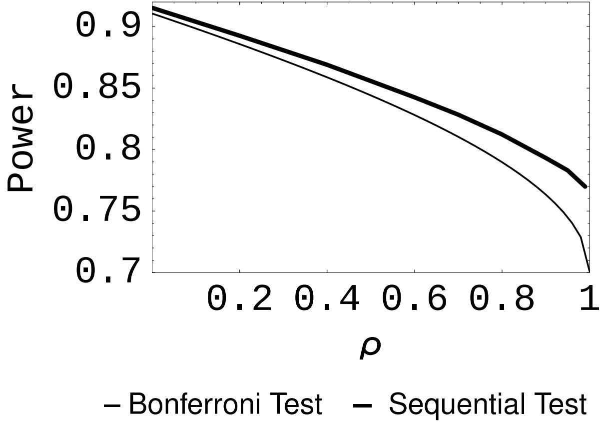
Type I Error Rates



$n = 275$
$\alpha = 0.05$

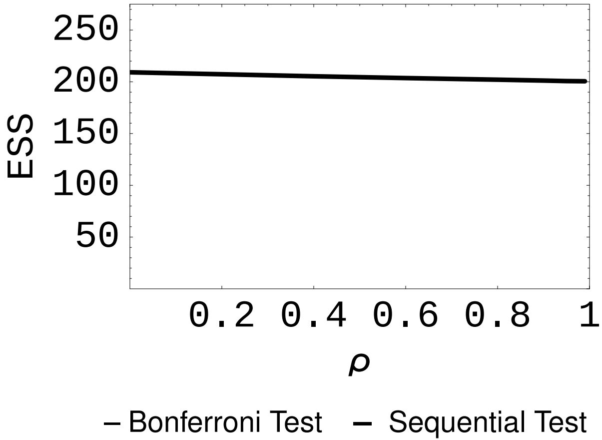
– Bonferroni Test **–** Sequential Test

Power to reject $H_A \cap H_B$



$n = 275$
$\alpha = 0.05$
$\mu_1 = 0.15\,\sigma$
$\mu_2 = 0.15\,\sigma$

Expected Sample Size



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$\alpha = 0.05$
$\mu_1 = 0.15\,\sigma$
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Some Comments

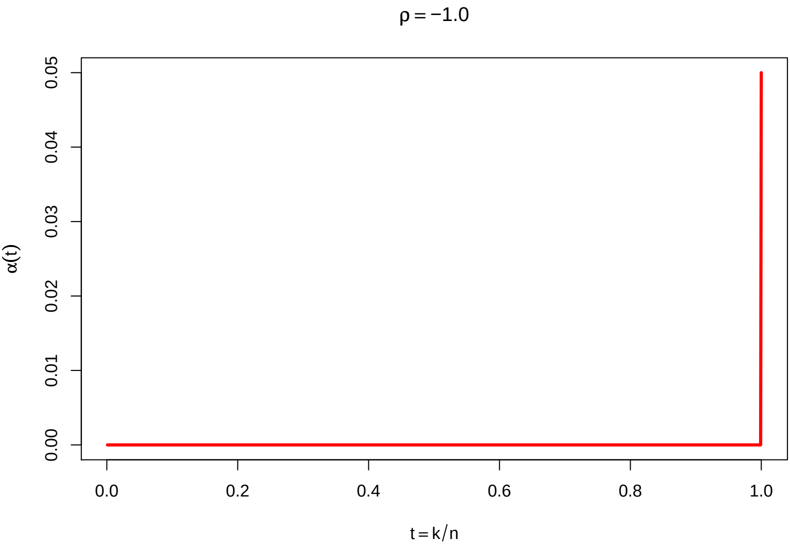
- The sequential test rejects the intersection hypothesis after the k-th observation, if

$$\sum_{i=1}^{k}(X_i^A + X_i^B) \geq 2\sqrt{n}z_{1-\alpha/2}$$

- The asymptotic α spending function is ($t = k/n$)

$$\alpha(t) = 2\left[1 - \Phi\left(\frac{\sqrt{2}z_{1-\alpha/2}}{\sqrt{t\,(1-\rho)}}\right)\right]$$

The Asymptotic α-Spending Function



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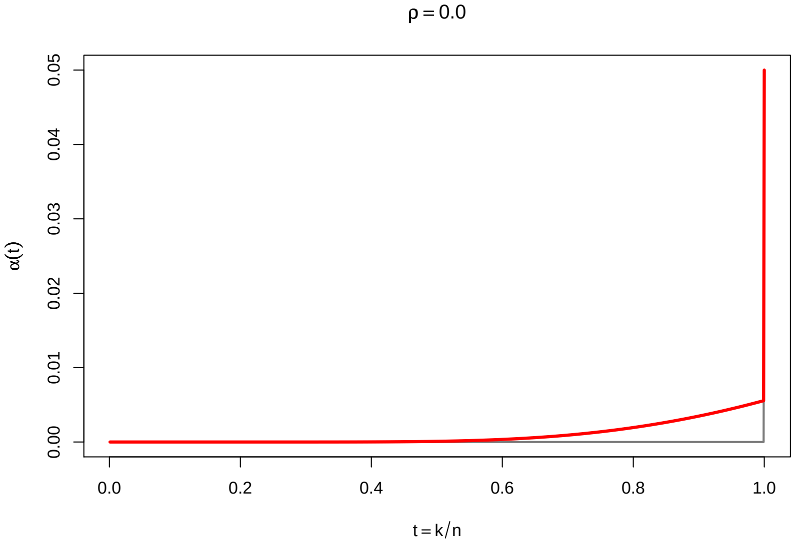

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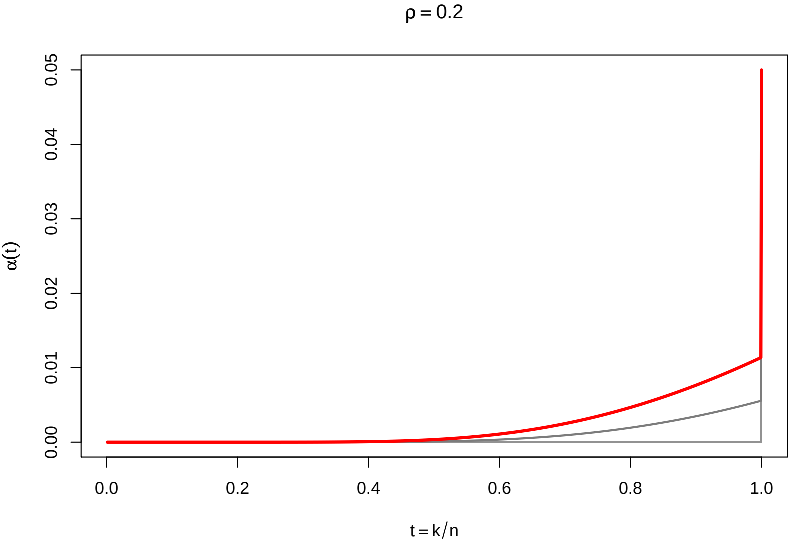

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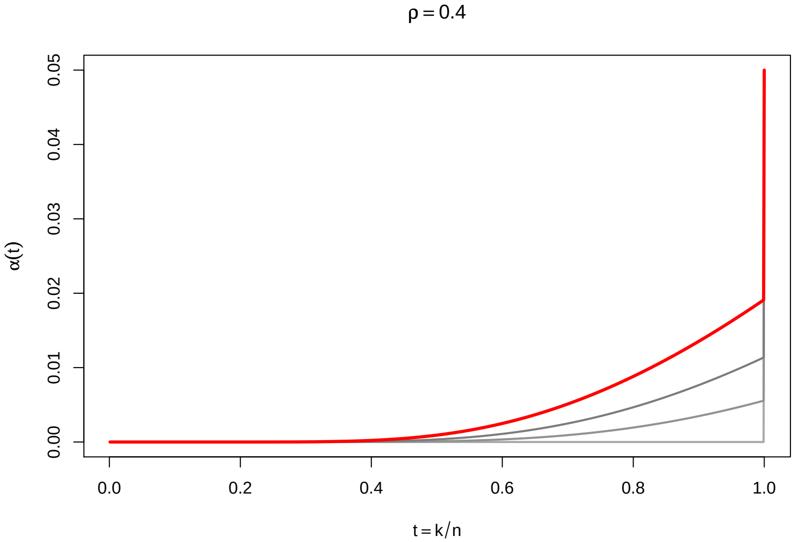

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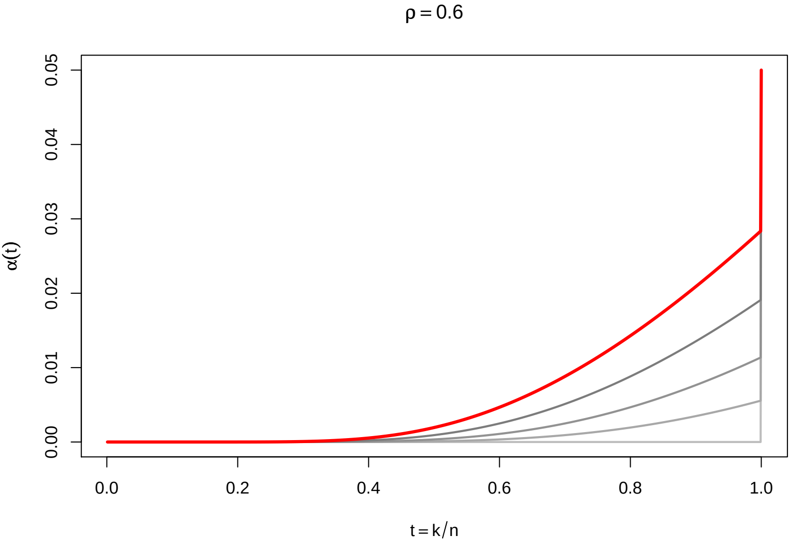

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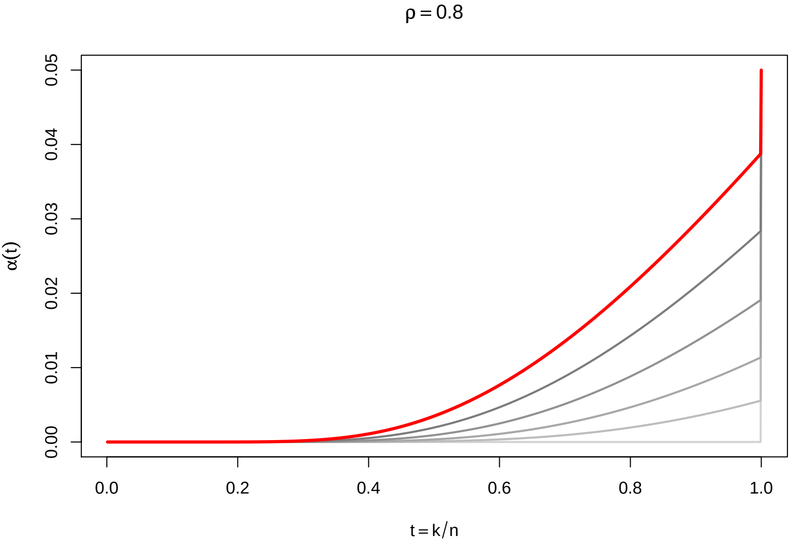

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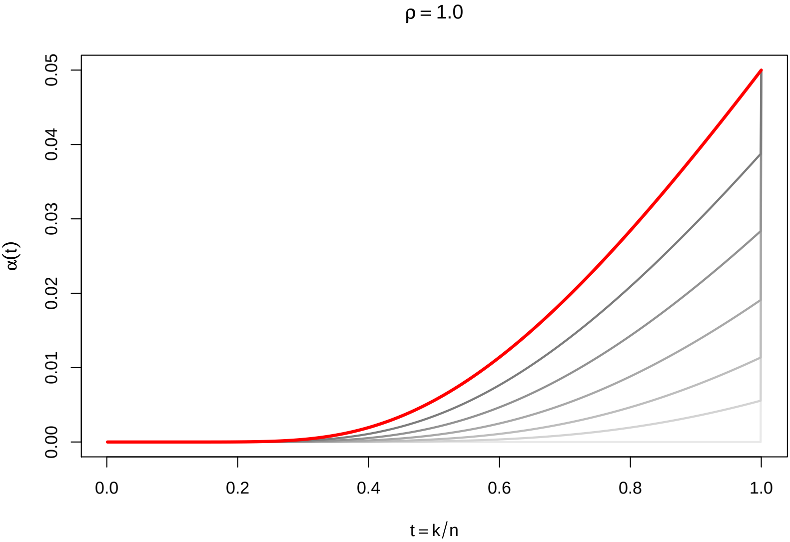

Generalizations

- Tests for elementary hypotheses with the closure principle
  - Test the intersection hypothesis with the sequential Bonferroni test
  - Test the elementary hypotheses with the sequential test defined by

$$2 \cdot \mathbf{1}_{\{p \leq \alpha/2\}}$$

- Weighted Bonferroni test for *m*-hypotheses
- General cut-off tests (Röhmel & Streitberg, 1987)

Adaptive Sequential Tests

- Truncated sequential test defined by

$$f : \mathbb{R}^n \rightarrow \{0\} \cap [0, 1]$$

- interim analysis after k observations
- choose a sample size m and a secondary sequential trial defined by a function g

$$g : \mathbb{R}^m \rightarrow \{0\} \cap [0, 1], \quad E\{g(\mathbf{X}'_m)\} = E\{f(\mathbf{X}_n) | \mathbf{X}_k\}$$

where $\mathbf{X}'_m$ denotes the vector of future observations.

MÜLLER & SCHÄFER, 2004

Summary

- Simple construction principle for truncated sequential tests
- Incorporating futility stopping
- Uniform improvement of Bonferroni tests and other cut-off tests
- Easily extended to an adaptive test

Selected References