ESTIMATION OF TIME VARIABLE PARAMETERS OF MACROECONOMIC MODEL WITH RATIONAL EXPECTATION

Tonner J.¹, Vašíček O.¹, Štecha J.², Havlena V.²

¹Department of Applied Mathematics and Computer Science
Faculty of Economics and Administration
Masaryk University
²Department of Control Engineering
Faculty of Electrical Engineering
Czech Technical University

ROeS 2007, Sept. 9-13, 2007 held in Bern, Switzerland.

This work was supported by Czech Science Foundation grant 402/05/2172, MŠMT project Research centre 1M0524, and funding of specific research at ESF MU.
Main features
Introduction
Hansen model
Bootstrap filter
Description of results
Conclusions
Problem statement and model

- Time-variant parameters estimation by Bootstrap filter
- Dynamic Stochastic General Equilibrium Model with Rational Expectation

Why to estimate time-variant parameters?

- Pure econometric approach or capricious nature.
- Parameters drifting as a character of economic environment.
- Parameters drifting as a telltale of model misspecification.

Our approach coincides with the second way.
Hansen model

\[
\max E \sum_{t=0}^{\infty} \beta^t [\ln(C_t) - \gamma H_t]
\]

under four conditions.

- \( Y_t = A_t K_t^\theta (\eta^t H_t)^{1-\theta} \)
- \( A_t = A^{1-\rho} A_{t-1}^\rho e^{\epsilon_t} \) \( \epsilon \sim N(0, \sigma^2), \text{cov}(\epsilon_i, \epsilon_j) = 0, i \neq j. \)
- \( Y_t = C_t + I_t \)
- \( K_{t+1} = (1 - \delta) K_t + I_t \)
After optimization where the representative consumer chooses \( \{Y_t, C_t, I_t, H_t, K_{t+1}\}_{t=0}^{\infty} \) to maximize utility subject to the four conditions, the system attains the form

\[
\begin{align*}
y_t &= a_t + \theta k_t + (1 - \theta) h_t \\
a_t &= \rho a_{t-1} + \epsilon_t \\
\kappa y_t &= (\kappa - \theta \lambda) c_t + \theta \lambda i_t \\
\eta k_{t+1} &= (1 - \delta) k_t + \lambda i_t \\
c_t + h_t &= y_t \\
-\frac{\eta}{\beta} c_t &= -\frac{\eta}{\beta} E_t c_{t+1} + \kappa E_t y_{t+1} - \kappa k_{t+1}
\end{align*}
\]

where \( \kappa = \frac{\eta}{\beta} - 1 + \delta \) and \( \lambda = \eta - 1 + \delta \).

(For more details see Ireland, 2004, or Maley, 2004).
Hansen model

Or in matrix form

\[
G \begin{bmatrix}
    x_{1,t+1} \\
    E_t x_{2,t+1}
\end{bmatrix} = D \begin{bmatrix}
    x_{1,t} \\
    x_{2,t}
\end{bmatrix} + \begin{bmatrix}
    e_{t+1} \\
    0
\end{bmatrix}, \text{ where}
\]

\[
x_{1,t} = \begin{bmatrix}
    k_t \\
    a_t
\end{bmatrix}, \quad x_{2,t} = \begin{bmatrix}
    y_t \\
    i_t \\
    h_t \\
    c_t
\end{bmatrix}, \quad e_t = \begin{bmatrix}
    0 \\
    e_{t+1}
\end{bmatrix},
\]

\[
G = \begin{bmatrix}
    \eta & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    -\kappa & 0 & \kappa & 0 & 0 & -\frac{\eta}{\beta} & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
    1 - \delta & 0 & 0 & \lambda & 0 & 0 & 0 \\
    0 & \rho & 0 & 0 & 0 & 0 & 0 \\
    -\theta & -1 & 1 & 0 & \theta - 1 & 0 & 0 \\
    0 & 0 & \kappa & -\theta \lambda & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & -\theta \lambda & \theta \lambda - \kappa & 0 \\
    0 & 0 & 0 & 1 & 0 & -1 & -1
\end{bmatrix}.
\]
Using Generalized Shur Decomposition of matrices $D$ and $G$ we get $D = QSZ^T$ and $G = QTZ^T$. It is not difficult to show that resulting state model attains the form

$$x_{1,t+1} = \begin{bmatrix} k_{t+1} \\ a_{t+1} \end{bmatrix} = \Pi \begin{bmatrix} k_t \\ a_t \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{t+1} \end{bmatrix},$$

$$x_{2,t} = \begin{bmatrix} y_t \\ i_t \\ h_t \\ c_t \end{bmatrix} = U \begin{bmatrix} k_t \\ a_t \end{bmatrix},$$

where matrices $\Pi$ and $U$ are constructed from matrices $Z, S, T$:

$$\Pi = Z_{kl}S_{ll}^{-1}T_{ll}Z_{kl}^{-1}$$

$$U = Z_{jl}Z_{kl}^{-1}.$$ 

Indexes $j, k, l$ denote appropriate submatrices corresponding with a number of stable roots (Söderlind, Paul 2003).
Ireland appends also VAR model, so

\[
\begin{bmatrix}
    k_{t+1} \\
    a_{t+1} \\
    \vartheta_{y_{t+1}} \\
    \vartheta_{c_{t+1}} \\
    \vartheta_{h_{t+1}}
\end{bmatrix}
= \begin{bmatrix}
    \Pi & 0 \\
    0 & D
\end{bmatrix}
\begin{bmatrix}
    k_t \\
    a_t \\
    \vartheta_{y_t} \\
    \vartheta_{c_t} \\
    \vartheta_{h_t}
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    \epsilon_{t+1} \\
    \xi_{y_{t+1}} \\
    \xi_{c_{t+1}} \\
    \xi_{h_{t+1}}
\end{bmatrix},
\]

\[
x_{2,t} = \begin{bmatrix}
    y_t \\
    i_t \\
    h_t \\
    c_t
\end{bmatrix}
= \begin{bmatrix}
    U & I
\end{bmatrix}
\begin{bmatrix}
    k_t \\
    a_t \\
    \vartheta_{y_t} \\
    \vartheta_{c_t} \\
    \vartheta_{h_t}
\end{bmatrix},
\]

where matrix

\[
D = \begin{bmatrix}
    d_{yy} & d_{yc} & d_{yh} \\
    d_{cy} & d_{cc} & d_{ch} \\
    d_{hy} & d_{hc} & d_{hh}
\end{bmatrix}
\]

is matrix of correlation coefficients and \(I\) is identity matrix. (page 11) (page 13)
Let us have non-linear discrete time system

\[ x(t + 1) = f(x(t), u(t)) + v(t), \]
\[ y(t) = g(x(t), u(t)) + e(t), \]

where \( v(t) \sim p_v(v(t)) \) and \( e(t) \sim p_e(e(t)) \) are white noises. Let us have CPDF

\[
p(x(t + 1)|x(t), u(t)) = p_v(x(t + 1) - f(x(t), u(t)))
\]
\[
p(y(t)|x(t), u(t)) = p_e(y(t) - g(x(t), u(t))).
\]

Suppose we have the set of observed data

\[ D^{t-1} = \{u(1), y(1), \ldots, u(t - 1), y(t - 1)\}. \]
Suppose the prior CPDF \( p(x(t)|D^{t-1}) \) is represented by an \( N \)-dimensional sample set

\[
x^{(i)}(t|t-1) \sim p(x(t)|D^{t-1}), i = 1, \ldots, N.
\]

The samples can be considered as an empirical CPDF

\[
p_N(x(t)|D^{t-1}) = \sum_{i=1}^{N} w^{(i)}(t|t-1) \delta(x(t) - x^{(i)}(t|t-1))
\]

with uniform weights

\[
w^{(i)}(t|t-1) = \frac{1}{N}, i = 1, \ldots, N.
\]

Then the knowledge of the state described by \( p(x(t)|D^{t-1}) \)can be updated by the following steps:
Data update step: Using $p(x(t)|D^{t-1})$ and the output measurement $y(t)$ (7), $p(x(t)|D^t)$ is determined as

$$p(x(t)|D^t) = \frac{p(y(t)|x(t), u(t))}{p(y(t)|D^{t-1}, u(t))} p(x(t)|D^{t-1}),$$

or empirically as

$$p_N(x(t)|D^t) = \sum_{i=1}^{N} w^{(i)}(t|t) \delta(x(t) - x^{(i)}(t|t)),$$

where $x^{(i)}(t|t) = x^{(i)}(t|t - 1)$.

The posterior weights are given by normalized likelihood

$$w^{(i)}(t|t) = \frac{p(y(t)|x(i)(t|t - 1), u(t))}{\sum_{j=1}^{N} p(y(t)|x(j)(t|t - 1), u(t))},$$

for $i = 1, \ldots, N$. 
Bootstrap filter

Time update step is theoretically given by state transition equation

\[ p(x(t + 1)|x(t), D^t) = p(x(t + 1)|x(t), u(t)), \]

or empirically by importance resampling of \( p_N(x(t)|D^t) \) with additive noise \( v(t) \), i.e.

\[ x^{(i)}(t + 1|t) = f(x^{(j)}(t|t), u(t)) + v^{(i)}(t), \quad (7) \]

where \( x^{(j)}(t|t) \) is a sample drawn from \( p_N(x(t)|D^t) \) and \( v^{(i)}(t) \) is sample drawn from \( p(v(t)) \).

The predicted, filtered and smoothed estimates of the state can be obtained as a weighted sample mean

\[ \hat{x}_{MS}(t|t') = \sum_{i=1}^{N} w^{(i)}(t|t') x^{(i)}(t). \]

More detailed description can be found in Štecha and Havlena, 1995 or Havlena and Štecha, 1998 or Trnka, 2004.
The results of $p(x(1)|D^0)$ carried out by widely used software DYNARE (see Juillard) can be seen in the following table.

<table>
<thead>
<tr>
<th></th>
<th>prior mean</th>
<th>postmean</th>
<th>conf. interval-l</th>
<th>conf. interval-r</th>
<th>priorstd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.990</td>
<td>0.992</td>
<td>0.988</td>
<td>0.998</td>
<td>0.005</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.004</td>
<td>0.004</td>
<td>0.002</td>
<td>0.006</td>
<td>0.002</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.200</td>
<td>0.267</td>
<td>0.164</td>
<td>0.388</td>
<td>0.070</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.004</td>
<td>1.002</td>
<td>0.976</td>
<td>1.027</td>
<td>0.015</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>0.025</td>
<td>0.005</td>
<td>0.052</td>
<td>0.012</td>
</tr>
<tr>
<td>$A$</td>
<td>6.000</td>
<td>6.100</td>
<td>4.142</td>
<td>7.452</td>
<td>1.000</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.990</td>
<td>0.994</td>
<td>0.989</td>
<td>0.999</td>
<td>0.005</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.390</td>
<td>0.276</td>
<td>0.205</td>
<td>0.355</td>
<td>0.070</td>
</tr>
<tr>
<td>$d_{yy}$</td>
<td>0.980</td>
<td>0.981</td>
<td>0.971</td>
<td>0.987</td>
<td>0.005</td>
</tr>
<tr>
<td>$d_{yc}$</td>
<td>0.000</td>
<td>-0.067</td>
<td>-0.143</td>
<td>0.010</td>
<td>0.100</td>
</tr>
<tr>
<td>$d_{yh}$</td>
<td>0.000</td>
<td>0.126</td>
<td>0.033</td>
<td>0.224</td>
<td>0.100</td>
</tr>
<tr>
<td>$d_{cy}$</td>
<td>0.000</td>
<td>0.093</td>
<td>0.011</td>
<td>0.159</td>
<td>0.100</td>
</tr>
<tr>
<td>$d_{cc}$</td>
<td>0.980</td>
<td>0.979</td>
<td>0.961</td>
<td>0.993</td>
<td>0.010</td>
</tr>
<tr>
<td>$d_{ch}$</td>
<td>0.000</td>
<td>0.104</td>
<td>-0.002</td>
<td>0.222</td>
<td>0.100</td>
</tr>
<tr>
<td>$d_{hy}$</td>
<td>0.000</td>
<td>-0.407</td>
<td>-0.492</td>
<td>-0.312</td>
<td>0.100</td>
</tr>
<tr>
<td>$d_{hc}$</td>
<td>0.000</td>
<td>-0.268</td>
<td>-0.348</td>
<td>-0.171</td>
<td>0.100</td>
</tr>
<tr>
<td>$d_{hh}$</td>
<td>0.300</td>
<td>0.304</td>
<td>0.286</td>
<td>0.321</td>
<td>0.010</td>
</tr>
<tr>
<td>$v_y$</td>
<td>0.280</td>
<td>0.353</td>
<td>0.284</td>
<td>0.443</td>
<td>0.080</td>
</tr>
<tr>
<td>$v_c$</td>
<td>0.460</td>
<td>0.450</td>
<td>0.370</td>
<td>0.520</td>
<td>0.110</td>
</tr>
<tr>
<td>$v_h$</td>
<td>0.002</td>
<td>0.002</td>
<td>0.001</td>
<td>0.002</td>
<td>0.001</td>
</tr>
</tbody>
</table>
We carried out two experiments:

- time - variable parameters estimation with VAR model (page 7)
- time - variable parameters estimation without VAR model (page 6)
Time - variable parameters estimation with VAR model

Fig. I. Outputs $y$ - with VAR Fig. II. State $x$ - with VAR
Parameter $\theta$ was set time-variant.

Fig. III. Parameter $\theta$ - with VAR

Fig. IV. Parameter $\theta$ - with VAR
Description of results

Time-variant parameters estimation without VAR model

Fig. V. Outputs $y$ - without VAR Fig. VI. State $x$ - without VAR
Description of results

Fig. VII. Parameter $\theta$ - suppressed VAR
Fig. VIII. Parameter $\delta$ - suppressed VAR
Description of results

Fig. IX. Parameter $\rho$ - suppressed VAR

Fig. X. Outputs $y$ - suppressed VAR

Tonner J.¹, Vašíček O.¹, Štecha J.², Havlena V.²

Estimation of time variable parameters of macroeconomic model
The method of time-variant parameter estimation by Bootstrap filter can be applied only to models which are well-determined.

The Hansen model with VAR and with time-variant parameters is a typical example of overcomplete model.

On the other hand, the poor base causes that we are not able to sufficiently fit the data (experiment without VAR). In this case we tried to complete the base with the time-variant parameters and with the suppressed VAR model.

Our work also brings new theoretical contribution which embodies in solution of rational expectations problem for time-variant parameter estimation by Bootstrap filter.
Thank you for your attention

Related papers of the authors are available on:

jtonner@tiscali.cz

A few preprints/reprints available from the authors at this conference.