Plausible reasoning and graded information: a unified approach

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Abstract

We propose a logic formalising implicational relationships where an explicit numerical degree is used to express uncertainty. The referred properties are assumed to be crisp. The uncertainty about an implication "from α it follows β" is indicated by a value reflecting the implausibility of $\alpha \wedge \neg\beta$. Depending on how the implausibility of disjunctions is determined by the implausibility of its disjuncts, we arrive at Possibilistic Logic or a modification hereof.

We extend this calculus to include properties of the form that some vague criterion is fulfilled to a specific degree. Here vague properties are treated as parametrised sets of crisp properties, which are either modelled as mutually disjoint or as partially overlapping. Moreover, a rule is included to ensure smoothness of the implausibility value with regard to changes of the degrees to which the properties under consideration hold.

As a result we obtain a calculus dealing both with uncertainty and gradedness of information. The two aspects are treated completely independently, but can optionally be interconnected in a controlled way. The benefit of the approach is illustrated on the basis of an example from medical decision support.

1 Introduction

The difficult task to represent experience-based knowledge includes the necessity to account for two basic aspects: uncertainty and vagueness. For the formalisation of knowledge in fields like medicine these aspects are met inevitably. Automated decision support in clinical diagnosis requires a framework capable of representing information which is not necessarily fully established and flexible enough to cope with notions which are not under all circumstances clearly applicable.

Needless to say, a great portion of the conclusions drawn in medicine is endowed with uncertainty. In particular at the beginning of a diagnostic process it is rarely possible to decide about the presence of a specific disease; facts are reviewed according to availability and these facts often allow only a tentative diagnosis. Moreover, concepts used in medicine often involve vagueness. In particular, coarse concepts are commonly used to characterise situations which vary in a continuous manner. For example, a laboratory test applied to detect a certain pathological condition typically involves real parameters. It is however difficult to indicate sharp boundaries of the range of normal values; the statement that the measured value is in the normal range can not be delimited from its negation in a convincing way. The question if the pathological condition applies does not always allow a clear answer; apart from a clear "yes" and clear "no", only a tendency can be indicated in the borderline areas.

In this paper we propose a formalism which treats both uncertainty and vagueness. Uncertainty is understood as plausibility and our general framework is Dubois and Prade's Possibilistic Logic. Furthermore, we will not deal directly with the phenomenon of vagueness but we just formalise the gradedness of properties. In our concluding section we will indicate how the calculus can serve as a theoretical framework for those expert systems which, as it is frequently the case in the area of medicine, offer the possibility to process graded information.

The many facets of uncertainty and its formal treatment

The development of formal systems dealing with uncertainty is the subject of a lively research field. Numerous formal systems for reasoning under uncertainty have been proposed in the past and several branches have emerged. For a recent overview we refer to the comprehensive paper [DuPr2] and the references given there; among the introductory monographs we may mention [Par] and [Hal]. The picture is rather inhomogenous. The approaches are motivated by different types of applications and by different ways to understand uncertainty; a systematisation is not easy. We will try a few general remarks in order to put our own approach into the right context.

The presumably best-known approach to uncertainty, which by many people is even viewed as the canonical way to go, relies on probability theory. Probabilities may be based on objective experience or subjective assumptions. In the former case we may speak about definite uncertainty: according to the common frequentist interpretation we assert a specific proportion of positive outcomes when checking a certain property repeatedly in a specific context. In the latter case we evaluate ad hoc to which extent the available information fits to the possible answers to a question. It has been frequently argued that both cases, although apparently unrelated, are modelled appropriately by means of single probability distributions. For an overview of formalisms referring to the probabilistic framework, see [Hal]. For the involved discussion on subjective probability see for instance [KySm] and concerning the limits of probabilities in this context see especially [TvKa].

Probabilistic logics reach practical limits when not enough relevant data is available. Indeed, probabilities assign in a precise sense to each event its weight among the whole. If reasoning is to be based on objective grounds a possibly high amount of knowledge about the referred situation must be available; otherwise the benefit of the probabilistic approach decreases. Popular methods like Bayesian networks reduce this necessity by making certain independence assumptions; these aditional assumption can be reasonable in applications but need not.

As an example where these problems are present we may mention the medical expert system CADIAG-2 [AdKo]. Its knowledge-base consists of weighted implicational relationships and the weights are understood as degrees of evidence. In [Pic], the attempt was made to reinterpret the CADIAG-2 knowledge base and to base the system on probabilistic grounds. It turned out that a performance comparable to the original system could be reached with probabilistic methods only under strong additional assumptions.

For situations in which the amount of information is not sufficient to reason on the firm grounds of probability alternative ways may be considered. First of all, approaches exist to cope with probability in situations where information is limited; we have to mention in particular the formalisms based on imprecise probabilities [Wal]. Another, in fact much more modest concern is to describe the effect of insufficient knowledge alone; in this case we quantify beliefs in facts on purely subjective grounds and formalise our inability to describe a situation objectively. Our present contribution is to be understood as referring to this restricted framework.

To model insufficient knowledge turns out to be an issue rather different from what is addressed by probabilistic theories. Recall that in the probabilistic framework all events as well as their negations, provided that their probabilities are neither 0 nor 1, are viewed as possible; we reason about what is known to possibly occur. Our concern is to argue about this possibility itself and to leave frequencies of occurrence apart: we may argue about the question if possibly already decided facts hold or not and quantify the degree of belief in it.

In spite of the different nature of problems one might certainly be tempted to apply once again probability theory. For instance, one may assign the probability

$\frac{1}{n}$ to each of n alternatives about which it is just known that one of them is true. This procedure is however inappropriate; we are led into trouble if one of these possibilities splits up to two alternatives which due to our ignorance are both not less probable than the remaining $n - 1$ ones. The limits of probability theory in situations characterised by lack of knowledge have been often discussed; see, for instance, [Sha, DuPr2].

An approach to model lack of knowledge

To deal with lack of knowledge, quite a range of formalism exists. The usual approach is to describe the extent of ignorance about considered situations by a numerical value. Once again, it is certainly understood that these numerical values are now supposed to reflect the degree of uncertainty if a property is verified if checked, rather than providing information about the proportions of occurrence and non-occurrence.

There is presumably at present no way to put a notion of quantified ignorance onto similarly firm grounds as in the case of probability theory. For this reason the divergence of formalisms which come into question for our concern is not surprising. The article [FrHa] offers some systematisation. To this end the notion of a plausibility space is introduced reflecting the minimal structural needs: a Boolean algebra $\mathcal{B}$ modelling some collection of properties, together with a partial order $\preccurlyeq$ extending the partial order $\leq$ of $\mathcal{B}$. For some $A, B \in \mathcal{B}$, $A \preccurlyeq B$ is supposed to mean that A is less plausible than B. In particular, if $\preccurlyeq$ is a linear order it can be replaced by a function from $\mathcal{B}$ to some bounded linear scale. In practice then, the latter can be taken the real unit interval or a subset of it. Examples are the Dempster-Shafer belief functions [Sha] and Goldschmidt and Pearl's ordinal rankings [GoPe].

The setting provided by plausibility spaces alone is appealing because it restricts to the essential and hardly questionable structure needed to deal with uncertainty. To define on its basis a calculus in which there is, for instance, a reasonable analog of the modus ponens seems to be difficult though. The problem is that given the plausibility degrees of two mutually exclusive properties, there is no way to determine how plausible the disjunction is.

In this paper, we are guided by the following considerations. We assume to be in a situation where we do not have the possibility to tell about the truth or falsity of certain facts; we treat uncertainty as ignorance about facts where it is left open if these facts generally hold, sometimes hold, or generally not hold. It is furthermore our opinion that the very idea to quantify ignorance numerically is not itself a source of problems but simply mirrors the fact that we can be uncertain about something to a smaller or larger extent. So we do not question the standard way to model uncertainty: by means of real values between 0 and 1.

In order to model uncertainty in the indicated sense there are, however, further decisions to be made which are less easily justifiable. Logic deals with implicational relationships. But given the assertion that some property α implies another property β, in which way should we associate a degree of uncertainty to this claim? Furthermore, how do uncertainty degrees of several statements relate to the uncertainty of combined statements?

The syntactic objects of our calculus will have the form of implications endowed with an uncertainty degree, like $\alpha \stackrel{t}{\Rightarrow} \beta$. Our basic assumption about the assignment of degree of uncertainty is as follows: to conclude from α to β is the more plausible the less plausible the situation specified by $\alpha \wedge \neg\beta$ is. As a technical consequence, we will in this paper actually not work with degrees of plausibility, but dually with degrees of "implausibility" or "surprise"; cf. [GoPe]. So t quantifies the degree of implausibility of $\alpha \wedge \neg\beta$. Using the truth constants, we can also refer to single propositions. Namely, $\top$ denoting the constant "true", the implication $\top \stackrel{t}{\Rightarrow} \alpha$ means that $\neg\alpha$ is implausible to the degree t.

Next we have to decide how to assign plausibility degrees for disjoint properties. We let the plausibilities of conjunctions be lower bounded by a function combining the plausibilities of the disjuncts. As a combining function we allow to use any fixed t-norm. We have in mind, however, to use only the three major examples of continuous t-norms, the Łukasiewicz, product, and Gödel t-norm. In the first two cases we are led to (the dual analog of) subadditive measures. In the third case we arrive at possibility measures; the plausibility of $\alpha \vee \beta$ is in this case uniquely determined as the larger value among the plausibilities of α and β. Consequently we are led to Possibilistic Logic then. Possibilistic Logic was introduced by Dubois and Prade and has been intensively developed since then; for an overview see, e.g., [DuPr1]. Moreover, an axiomatisation of first-order Possibilistic Logic is presented by J. Lang in [Lan]. Lang's calculus is, for the choice of the Gödel t-norm, formally equivalent to ours.

Vague properties: treated as continuous sets of crisp properties

Having a strong calculus to deal with uncertainty at hand, the aim of the paper is to incorporate vague properties. Vague properties have been dealt with in the framework of Possibilistic Logic several times; see, for instance, [FGM, DGM]. Our approach however is specific in some points.

Vague properties involve two levels of perception, a coarse one (for instance, distinguishing “small”, “average-sized”, and “large”) which defines the property (like “large” as observably distinguished from “average-sized”) and a fine one (for instance, the positive rationals) which is the result of an iterative process reflecting the underlying intuition (like “size”). The coarse concepts can be represented by means of the fine structure as a fuzzy set. We will follow this standard method and we will use the operations of pointwise minimum, maximum, and standard negation to build new fuzzy sets from given ones.

However, we do not intend to formalise relationships involving vague properties themselves, as represented by fuzzy sets. Possibilistic reasoning about “properly” vague properties is the subject of the articles [FGM] and [DGM], where possibilistic measures are generalised from Boolean algebras to MV-algebras and Gödel algebras of fuzzy sets, respectively. Here, in contrast, we will not generalise Dubois and Prade’s original method. We will rather consider statements of the form that a vague property holds to a specific degree; these statements are crisp. So if φ denotes a vague property, we will consider the parametrised set (φ, t) of properties, meaning that φ holds to the degree t where t is an element of the real unit interval.

Note that this set is a collection of mutually exclusive and exhaustive crisp properties. In a first approach, we will model (φ, t), where t varies over the rationals between 0 and 1, exactly in this way. However this approach causes problem with regard to a proper interpretation, apart from inconveniences which also exist in mathematical respects. If (φ, t) is supposed to be a statement reflecting somebody’s observation or impression, infinity seems inappropriate. One solution is easy: we may restrict the number of degrees of presence to finitely many ones. For the case that it is intended to model the transition between, say, $(\varphi, 0)$ and $(\varphi, 1)$ in a continuous way, we offer a modified approach. In our alternative model (φ, t) is interpreted in the same way as the conjunction of (φ, s), where $t - \zeta < s < t + \zeta$ and $\zeta > 0$ is fixed, in the prior model. The motivation is not to account for ignorance about the degrees, which are anyhow determined ad hoc. The motivation is, intuitively speaking, to account for the idea to model each statement by an extended region in a continuous space rather than by a point-like one.

To motivate the last topic discussed in this paper, suppose that we are certain to the degree d that if φ fully applies some further (crisp) property α is true as well. In the framework as mentioned so far, nothing could be said, say, with regard to $(\varphi, 0.9)$. To reduce this drawback, we strengthen the logic. We add a rule which amounts to Lipschitz continuity: we require that changes of the degree of uncertainty are bounded by changes of the degree to which φ holds. So for example, we may derive that $(\varphi, 0.9)$ still implies α where however the degree of uncertainty is reduced to $d - \tau 0.1$. Here, $\tau > 0$ is a fixed parameter controlling the strength of the smoothness rule.

An application in medicine

The medical expert systems of the CADIAG family have been developed under the supervision of K.-P. Adlassnig at the University of Vienna Medical School, now the Medical University of Vienna, since the 1980's [AdKo]. They offer clinical decision support for instance in the field of rheumatological diseases. Roughly speaking, from a given pattern of clinical symptoms and signs of a patient the possible diagnoses are derived. CADIAG-2 is distinguished from other expert systems in that it allows the input to be graded; for instance, a symptom may be endowed with a real number between 0 and 1 according to the degree to which it is present. Furthermore, the output is a set of weighted possibilities; each weight is the degree of certainty associated with a disease.

The performance of CADIAG-2 is convincing, as demonstrated, e.g., in [LAK]. A formally oriented framework based on transparent principles is still a research topic though. In particular, the systems contains aspects of both fuzzy logic and uncertainty reasoning without a clear separation. To put the inference method of CADIAG-2 on firm semantic grounds is thus quite a challenge, which has recently been the subject of several papers. In [CiVe] the fuzzy logical aspects were studied and it turned out that fuzzy logic alone is not sufficient as a basis of CADIAG-2. In contrast, [Pic] examines aspects of uncertainty and is in fact written from the probabilistic point of view, whereas the aspect of vagueness is not taken into account. The present paper seeks to account in a transparent way for both uncertainty and vagueness. As we will indicate by an example at the end of this paper, the formalism described in this paper may bring us one step closer to a satisfactory solution of the problem how to embed CADIAG-2 into a conceptually clear framework.

We have actually developed our calculus in view of this application. However we should stress that we have worked out the details independently from any specific application. In particular, we will not justify any of the inavoidable "design choices" by reference to the needs of medical expert systems. If we had done so, we would have actually decreased the value of the formalism for the intended application.

The paper is organised as follows. In the next Section 2, we introduce our framework, which is a slight generalisation of Possibilistic Logic. In fact, only Possibilistic Logic itself will play a role in the remainder of the paper. In the following Section 3, we extend the calculus so as to include reasoning with graded vague properties, whose model will be refined in Section 4. The mentioned rule for smoothness of the uncertainty degree is introduced in Section 5. Finally, we outline in the concluding Section 6 the practical application of our formalism in medicine.

2 Generalised possibilistic logic

In this section, we shall propose a generalised version of Possibilistic Logic as a logic for reasoning under uncertainty. According to the considerations of the introductory chapter about the nature of statements which we are going to formalise, we shall first specify our framework informally and formally.

As usual we distinguish the content level and the belief level. On the content level, we refer to the factual content of our reasoning. This is a set of situations about which we assume that exactly one of it always holds. We distinguish between these situations classically: we choose a set of yes-no properties; each property holds in a given situation or not; and each considered situation is uniquely determined by knowing which properties hold and which do not hold in it. So in particular, a property is identified with a subset of a fixed set of situations.

We do not assume that a property can be checked to hold or not to hold in any situation. This gives rise to a second level, called the belief level. Here the subjective sphere of the "agent", that is, the one who reasons about the given set of situations, comes into play. Namely, we allow statements about the mutual relationship of properties which rely on possibly nonconclusive experience. That is, we allow to take into account knowledge about relationships between properties even if this knowledge is speculative. If $\alpha_1, \ldots, \alpha_k, \beta$ denote properties, a typical statement will look like

$$\alpha_1, \ldots, \alpha_k \stackrel{d}{\Rightarrow} \beta; \tag{1}$$

here d is an element of the real unit interval expressing the agent's ignorance in a quantitive way. Starting from $d = 1$ expressing certainty, we may decrease d to express a reduced confidence in the correctness of the implication (1), which for $d = 0$ is true in any case.

Note that, at least in principle, the value d in (1) can always be provided. Whereas the assignment of probabilities to properties means comprehensive knowledge about the situation, a degree of certainty can naturally be assigned under all circumstances; in the worst case, we put $d = 0$.

Let's now outline our formal framework. There is nothing special on the content level. As usual, situations are modelled by an unstructured set S and each property by a subset of S. We use the usual connectives, namely conjunction, disjunction, and negation; and we will include the always false and always true property as well. We note that the classical implication plays no role. The interpretation of the connectives is standard. In short, the collection of properties is modelled by a Boolean algebra $\mathcal{B}$ of subsets of S.

To see how we proceed on the belief level, consider two elements $A, B \in \mathcal{B}$ and let us identify them with the two properties which they model. The degree of uncertainty about the question if we can conclude B from A should depend in some way on numerical values associated to the elements of the Boolean subalgebra generated by A and B. In contrast to the probabilistic approach, where we have a measure on $\mathcal{B}$ and take the quotient of the values associated to $A \wedge B$ and A, we follow here a much simpler way, adopting the concepts of Possibilistic Logic. Namely, we will use only one element of the subalgebra, namely $A \wedge \neg B$, to which we associate a "degree of surprise" $d = \varrho(A \wedge \neg B)$. The larger d is, the less plausible is a situation in which A holds and B does not hold. This will be our interpretation of the statement that A implies B to the degree d.

We may understand this degree of implausibility also "positively", namely as a degree of certainty, in the straightforward way. To say that A implies B to the degree 1 means that A implies B; that is, 1 expresses full certainty. Similarly, to say that A implies B to the degree 0 means not to say anything; this relationship holds between an arbitrary pair of properties. The remaining values refer to a smaller or larger degree of certainty, and specific values of certainty strictly in between 0 and 1 refer to subjectively quantified amount of certainty.

So our basic model consists of the Boolean algebra $\mathcal{B}$ together with a mapping ϱ from $\mathcal{B}$ to the real unit interval $[0, 1]$. We call ϱ here a rejection function; with regard to the setting of [DuPr1], ϱ plays the role of a necessity measure (or alternatively of a possibility measure). If $A \in \mathcal{B}$, $\varrho(A)$ expresses the degree to which the property modelled by A would be found surprising if found to hold. We assume ϱ to be order-reversing and $\varrho(1) = 0$. Furthermore we assume that $\mathcal{B}$ does not model situations which are considered as definitely impossible; so we require that $\varrho(A) = 1$ holds exactly if $A = 0$. Furthermore, a property may consist of alternatives, say, we may have $A = B \vee C$. In this case, we allow the assumption that A is not more surprising than B or C and also not less than both; then $\varrho(A) = \min \{\varrho(B), \varrho(C)\}$. However, we also allow to use alternative combining function as long as it is fixed; our actual assumption about ϱ is that $\varrho(A) \geq \varrho(B) \odot \varrho(C)$ for some fixed t-norm $\odot$.

Let us fix a t-norm $\odot : [0, 1] \times [0, 1] \to [0, 1]$. For the notion of a t-norm and further detailed information on this operation, see, e.g., [KMP]. Furthermore, we will denote the operations minimum, maximum, and standard negation on $[0,1]$ by $\wedge$, $\vee$, $\sim$, respectively.

Definition 2.1. Let $(\mathcal{B}; \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. A *rejection function* on $\mathcal{B}$ w.r.t. $\odot$ is a mapping $\varrho\colon\ \mathcal{B} \to [0,1]$ such that, for any $A, B \in \mathcal{B}$, (i) $\varrho(1) = 0$, (ii) $\varrho(A) = 1$ if and only if $A = 0$, (iii) $A \leq B$ implies $\varrho(B) \leq \varrho(A)$, and (iv) we have

$$\varrho(A \vee B) \ \geq\ \varrho(A) \odot \varrho(B).$$

A pair $(\mathcal{B}, \varrho)$ of a Boolean algebra $\mathcal{B}$ and a rejection function ϱ on $\mathcal{B}$ will be called a *Boolean uncertainty algebra*.

Let us consider the case that $\odot = \wedge$; this choice for $\odot$ will actually be predominant in our paper. Then condition (iv) can be formulated as

$$\varrho(A \vee B) \ =\ \varrho(A) \wedge \varrho(B), \tag{2}$$

where we have made use of the antitonicity of ϱ. It further follows that

$$N\colon\ \mathcal{B} \to [0,1], \quad A \mapsto \varrho(\neg A)$$

is a necessity measure – see [DuPr1] –, and our logic will turn out to be equivalent with Possibilistic Logic.

We proceed with the model-theoretic definition of what we call Generalised Possibilistic Logic, denoted by $\mathtt{I}^{\odot}$, where the "$\mathtt{I}$" stands for "ignorance". The choice of this name is motivated by the fact that $\mathtt{I}^{\wedge}$ is essentially Possibilistic Logic and $\mathtt{I}^{\odot}$ is a straightforward generalisation of $\mathtt{I}^{\wedge}$. In this paper, $\mathtt{I}^{\wedge}$ is still most important, and we will write in the sequel $\mathtt{I}$ instead of $\mathtt{I}^{\wedge}$.

Our language will be finite; let's fix a number $N \geq 1$ of variable symbols. Several results in the sequel would remain the same if we allowed a countably infinite set of variables; however, we do not see an important reason to do so.

Definition 2.2. The *propositions* of $\mathtt{I}^{\odot}$ are built up from *variables* $\varphi_1, \ldots, \varphi_N$ and the *truth constants* $\bot, \top$ by means of the binary connectives $\wedge, \vee$ and the unary connective $\neg$. We will denote the set of propositions by $\mathcal{P}$.

An *implication* of $\mathtt{I}^{\odot}$ is a triple consisting of a finite non-empty set of propositions $\alpha_1, ..., \alpha_k$, a proposition β, and an element d of the real unit interval, denoted $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$. Here $\alpha_1, ..., \alpha_k$ are called the *antecedents*, β is the *succedent*, and d is the *degree of certainty*.

An *evaluation* for $\mathtt{I}^{\odot}$ is a mapping v from $\mathcal{P}$ to a Boolean uncertainty algebra $(\mathcal{B}, \varrho)$ such that $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta),\ v(\alpha \vee \beta) = v(\alpha) \vee v(\beta),\ v(\neg\alpha) = \neg v(\alpha)$ for $\alpha, \beta \in \mathcal{P}$ and $v(\bot) = 0,\ \ v(\top) = 1$. An implication $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$ is then *satisfied* by v if

$$\varrho(v(\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \neg\beta)) \ \geq\ d.$$

A *theory* is a set of implications. We say that a theory $\mathcal{T}$ *semantically entails* an implication $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$ if, for all evaluations v, whenever all elements of $\mathcal{T}$ are satisfied by v, then also $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$ is satisfied by v.

We axiomatise the logic $\mathtt{I}^{\odot}$ as follows. Here, rules are pairs of a possibly empty finite set of implications and one further implication. The Greek Γ denotes a finite set of antecedents, and as usual, expressions like Γ, α, β denote $\Gamma \cup \{\alpha, \beta\}$, where it is not assumed that these sets must not overlap or that α and β must be distinct. The set Γ can be empty; recall however that an implication has at least one antecedent; thus in an expression like $\Gamma \stackrel{t}{\Rightarrow} \alpha$, Γ must be non-empty.

For the case that $\odot = \wedge$ the following calculus is the propositional part of Lang's calculus in [Lan], just presented in a modified way. In particular, instead of a necessity measure, we use the complemented possibility measure.

Definition 2.3. The following are the rules of $\mathtt{I}^{\odot}$, where α, β, γ are propositions, Γ is a finite set of propositions, and $c, d \in [0, 1]$:

$$\bot \stackrel{d}{\Rightarrow} \alpha \qquad \alpha \stackrel{d}{\Rightarrow} \alpha \qquad \alpha \stackrel{d}{\Rightarrow} \top \qquad \alpha, \neg\alpha \stackrel{d}{\Rightarrow} \bot$$

$$\alpha \stackrel{0}{\Rightarrow} \beta \qquad \frac{\Gamma \stackrel{d}{\Rightarrow} \alpha}{\Gamma \stackrel{c}{\Rightarrow} \alpha} \text{ where } c < d \qquad \frac{\Gamma \stackrel{c}{\Rightarrow} \alpha \quad \alpha \stackrel{d}{\Rightarrow} \beta}{\Gamma \stackrel{c \odot d}{\Rightarrow} \beta}$$

$$\frac{\Gamma \stackrel{d}{\Rightarrow} \alpha}{\Gamma, \beta \stackrel{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \alpha, \beta \stackrel{d}{\Rightarrow} \gamma}{\Gamma, \alpha \wedge \beta \stackrel{d}{\Rightarrow} \gamma} \qquad \frac{\Gamma \stackrel{c}{\Rightarrow} \alpha \quad \Gamma \stackrel{d}{\Rightarrow} \beta}{\Gamma \stackrel{c \odot d}{\Rightarrow} \alpha \wedge \beta}$$

$$\frac{\Gamma, \alpha \stackrel{c}{\Rightarrow} \gamma \quad \Gamma, \beta \stackrel{d}{\Rightarrow} \gamma}{\Gamma, \alpha \vee \beta \stackrel{c \odot d}{\Rightarrow} \gamma} \qquad \frac{\Gamma \stackrel{d}{\Rightarrow} \alpha}{\Gamma \stackrel{d}{\Rightarrow} \alpha \vee \beta} \qquad \frac{\Gamma \stackrel{d}{\Rightarrow} \beta}{\Gamma \stackrel{d}{\Rightarrow} \alpha \vee \beta}$$

$$\frac{\alpha \stackrel{d}{\Rightarrow} \beta}{\neg\beta \stackrel{d}{\Rightarrow} \neg\alpha} \qquad \frac{\neg\alpha \stackrel{d}{\Rightarrow} \beta}{\neg\beta \stackrel{d}{\Rightarrow} \alpha} \qquad \frac{\alpha \stackrel{d}{\Rightarrow} \neg\beta}{\beta \stackrel{d}{\Rightarrow} \neg\alpha}$$

The notion of a proof of an implication $\alpha \stackrel{d}{\Rightarrow} \beta$ from a theory $\mathcal{T}$ is defined in the usual way. We write $\mathcal{T} \vdash \alpha \stackrel{d}{\Rightarrow} \beta$ if there exists one.

A theory $\mathcal{T}$ is called *consistent* if $\mathcal{T} \vdash \top \stackrel{d}{\Rightarrow} \bot$ implies $d = 0$.

The proof of the completeness Theorem 2.7 for $\mathtt{I}^{\odot}$ below is possible along routine lines; we calculate the Lindenbaum algebra associated to a given theory, and the maximal d such that the theory proves $\alpha \stackrel{d}{\Rightarrow} \bot$ is taken as the value to which ϱ maps the equivalence class of α. For $\mathtt{I}$, a proof is moreover contained in [Lan].

In spite of this, we will devote the remainder of this section to present a fully detailed proof which is even more involved than necessary. We do so because in the subsequent sections, we will present three logics which are successively more special than $\mathtt{I}^{\odot}$; we shall proceed then in full analogy to the easy case discussed here.

For propositions α and β, we write $\alpha \rightarrow \beta$ to abbreviate $\neg\alpha \vee \beta$.

Lemma 2.4. *Let α, β be propositions of $\mathtt{I}^{\odot}$. Then $\vdash \alpha \stackrel{1}{\Rightarrow} \beta$ if and only if $\alpha \rightarrow \beta$ is a tautology of classical propositional logic.*

Proof. Let $\mathtt{I}^{\odot}$ prove $\alpha \stackrel{1}{\Rightarrow} \beta$. Since the degree associated to the conclusion is in each rule smaller or equal to each degree in the assertions, a proof $\alpha \stackrel{1}{\Rightarrow} \beta$ can be assumed to involve the degree 1 only. It follows $\alpha \rightarrow \beta$ is a classical tautology.

The converse assertion is easily checked. □

For some set Ω, let us denote the Boolean algebra of subsets of Ω by $P\Omega$. For instance, $P\{0,1\}$ is the two-element Boolean algebra.

Furthermore, $P\{0,1\}^N$ denotes the free Boolean algebra with N generators. We will identify the latter with the Boolean algebra of subsets of $\{0,1\}^N$, that is, with $P(\{0,1\}^N)$.

We will now consider the Lindenbaum algebra associated to $\mathtt{I}^{\odot}$, where equivalence of propositions will mean that one implies the other one to the degree 1. Accordingly, the somewhat loose statements "α implies β" and "α and β are equivalent" mean that $\mathtt{I}^{\odot}$ proves $\alpha \stackrel{1}{\Rightarrow} \beta$, or both $\alpha \stackrel{1}{\Rightarrow} \beta$ and $\beta \stackrel{1}{\Rightarrow} \alpha$, respectively. Analogous remarks apply to all logics considered in subsequent sections as well.

Lemma 2.5. *For propositions α and β of $\mathtt{I}^{\odot}$, we put $\alpha \approx \beta$ if $\mathtt{IG}^{\zeta}$ proves $\alpha \stackrel{1}{\Rightarrow} \beta$ and $\beta \stackrel{1}{\Rightarrow} \alpha$. Then the quotient $\langle\mathcal{P}\rangle$ of $\mathcal{P}$ w.r.t. $\approx$, endowed with the induced operations $\wedge, \vee, \neg$ and the constants $\langle\bot\rangle, \langle\top\rangle$, is a Boolean algebra isomorphic to $\bar{\mathcal{B}} = P\{0,1\}^N$. The isomorphism is given by*

$$w(\langle\varphi_i\rangle) \;=\; \{(r_1, \ldots, r_N) \in \{0,1\}^N : \, r_i = 1\}, \quad i = 1, \ldots, N$$

Furthermore, let $\varrho : \; \bar{\mathcal{B}} \rightarrow [0,1]$ be 0 on all non-zero elements; then $(\bar{\mathcal{B}}, \varrho)$ is a Boolean uncertainty algebra.

Define $\bar{v}\colon \mathcal{P} \to \{0,1\}^N,\ \ \alpha \mapsto w(\langle\alpha\rangle)$. *Then* $\bar{v}$ *is an evaluation of* $\mathtt{IG}^\zeta$ *such that* $\bar{v}(\alpha) = \emptyset$ *if and only if* $\vdash \alpha \overset{1}{\Rightarrow} \bot$.

Proof. By Lemma 2.4, $\langle\mathcal{P}\rangle$ is the free Boolean algebra with N generators.

Clearly, $(\bar{\mathcal{B}}, \varrho)$ is a Boolean uncertainty algebra and $\bar{v}$ is an evaluation for $\mathtt{I}^\odot$. Furthermore, $\mathtt{I}^\odot$ proves $\alpha \overset{1}{\Rightarrow} \bot$ iff $\langle\alpha\rangle = \langle\bot\rangle$ iff $w(\langle\alpha\rangle) = w(\langle\bot\rangle)$. Given that $\bar{v}(\alpha) = w(\langle\alpha\rangle)$ and $w(\langle\bot\rangle) = \emptyset$, the last part follows. □

In the next lemma, two theories proving the same implications are called equivalent.

Lemma 2.6. *Let* $\mathcal{T}$ *be a theory of* $\mathtt{I}^\odot$. *Then there is a theory* $\mathcal{T}'$ *which is equivalent to* $\mathcal{T}$ *and consists of*

$$\chi_0 \overset{d_0}{\Rightarrow} \bot,\ \ \chi_1 \overset{d_1}{\Rightarrow} \bot,\ \ \ldots,\ \ \chi_m \overset{d_m}{\Rightarrow} \bot,$$

where $\vdash \chi_i \wedge \chi_j \overset{1}{\Rightarrow} \bot$ *for* $i \neq j$, $\vdash \top \overset{1}{\Rightarrow} \chi_0 \vee \ldots \vee \chi_m$, *and* $1 = d_0 > d_1 \geq \ldots \geq d_{m-1} > d_m = 0$.

In case $\odot = \wedge$ *we may require* $d_0 > d_1 > \ldots > d_m$.

Proof. We claim that $\mathcal{T}$ is equivalent to a finite theory. Indeed, any proposition appearing in $\mathcal{T}$ can be substituted by any equivalent one. Furthermore, by Lemma 2.5 there are, up to equivalence, only finite many propositions. Finally, if there are two implications in $\mathcal{T}$ differing only in the degree of certainty, the implication with the lower degree of certainty can be dropped.

Next, using Lemma 2.4, it is not difficult to see that $\alpha_1, ..., \alpha_k \overset{d}{\Rightarrow} \beta$ and $\alpha_1 \wedge \ldots \alpha_k \wedge \neg\beta \overset{d}{\Rightarrow} \bot$ are in $\mathtt{I}^\odot$ mutually derivable. So we assume that $\mathcal{T}$ contains only implications of the latter form.

If there are $\alpha_1 \overset{e_1}{\Rightarrow} \bot$ and $\alpha_2 \overset{e_2}{\Rightarrow} \bot$, where $e_1 \leq e_2$ and $\nvdash \alpha_1 \wedge \alpha_2 \overset{1}{\Rightarrow} \bot$, we may replace the first implication by $\alpha_1 \wedge \neg\alpha_2 \overset{e_1}{\Rightarrow} \bot$.

Finally, the implications with the degree 0 or 1 can be replaced by a single implication, combining the antecedents disjunctively. If $\odot = \wedge$, the same can be done with any implications whose degrees of certainty coincide. □

On the basis of these preliminaries we prove the completeness of $\mathtt{I}^\odot$.

Theorem 2.7. *Let* $\mathcal{T}$ *be a consistent theory of* $\mathtt{I}^\odot$ *and* $\Gamma \overset{e}{\Rightarrow} \delta$ *an implication of* $\mathtt{I}^\odot$. *Then* $\mathcal{T}$ *semantically entails* $\Gamma \overset{e}{\Rightarrow} \delta$ *if and only if* $\mathcal{T}$ *proves* $\Gamma \overset{e}{\Rightarrow} \delta$.

Proof. It is easily checked that all rules are sound. The "if" part follows.

By Lemma 2.6, we can assume that $\mathcal{T} = \{\chi_0 \overset{d_0}{\Rightarrow} \bot, \chi_1 \overset{d_1}{\Rightarrow} \bot, \ldots, \chi_m \overset{d_m}{\Rightarrow} \bot\}$, where the χ_i are pairwise disjoint and jointly exhaustive, and $1 = d_1 > d_2 \geq \ldots \geq d_{m-1} > d_m = 0$.

Let $\bar{v}\colon \mathcal{P} \to \bar{\mathcal{B}}$ be the evaluation according to Lemma 2.5. Let $S = \bar{v}(\neg\chi_0)$, and let $\mathcal{B} = [0, S]$ be endowed with the Boolean structure induced by $\bar{\mathcal{B}}$. Let

$$v\colon \mathcal{P} \to \mathcal{B}, \quad \alpha \mapsto \bar{v}(\alpha) \cap S.$$

We furthermore define

$$\varrho\colon \mathcal{B} \to [0,1], \quad A \mapsto \bigodot \{d_i\colon 1 \leq i \leq m \text{ and } A \wedge v(\chi_i) \neq 0\},$$

where the result is 1 in case the set is empty. This is obviously a rejection function on $\mathcal{B}$. Then v is an evaluation of $\mathtt{I}^{\odot}$ in the Boolean uncertainty algebra $(\mathcal{B}, \varrho)$. Since $\varrho(v(\chi_i)) = d_i$ for all i, all elements of $\mathcal{T}$ are satisfied by v.

Assume now that $\mathcal{T}$ does not prove $\Gamma \overset{e}{\Rightarrow} \delta$. So $\mathcal{T}$ does not prove $\alpha \overset{e}{\Rightarrow} \bot$, where α is the conjunction of $\Gamma \cup \{\neg\delta\}$. Let $d = \varrho(v(\alpha))$. We easily check that $\mathcal{T} \vdash \alpha \overset{d}{\Rightarrow} \bot$. It follows $d < e$, so in particular $\Gamma \overset{e}{\Rightarrow} \delta$ is not satisfied by v and $\mathcal{T}$ does not semantically entail $\Gamma \overset{e}{\Rightarrow} \delta$. The proof of the "only if" part is complete. ☐

We note that the model constructed in this proof is finite; so Theorem 2.7 could be reformulated to involve finite Boolean uncertainty algebras only.

3 Inclusion of graded properties: the finite case

We extend our framework to include properties which are not in every situation assumed to hold or not to hold. We still work with a Boolean algebra of subsets of some set S, and properties which, as it has been the case by now, are modelled by subsets of S will be called crisp. The variable symbols however will from now on denote vague properties and are interpreted by fuzzy sets over S. It will still be possible to reason about crisp properties because crispness is expressible in our framework.

It must be stressed that we will not allow statements of the form $\varphi \overset{d}{\Rightarrow} \psi$ where φ and ψ model vague properties. We will rather denote by (φ, s) the property that φ holds to the degree s, modelled by the subset of S consisting of those points which are mapped to s. The value s will be referred to as the degree of presence of φ

in such situations. We allow then statements of the form $(\varphi, s) \stackrel{d}{\Rightarrow} (\psi, t)$ and, as relationships between crisp properties, we will interpret them just as before.

We will furthermore allow vague properties to be combined by means of a conjunction $\wedge$, a disjunction $\vee$, and a negation $\sim$. These connectives are interpreted by the minimum, maximum, and standard negation applied pointwise to the respective fuzzy sets. For example, for some evaluation v, $(\varphi \vee \sim\psi, t)$ is interpreted by $\{a \in S\colon v(\varphi)(a) \vee \sim v(\psi)(a) = t\}$.

Our t-norm will from now on always be the Gödel t-norm, that is, we put $\odot = \wedge$. So in all what follows we stay in the realm of the Possibilistic Logic $\mathtt{I}$. The more general case remains to be explored.

Furthermore, for the moment we will restrict to a finite set of degrees of presence: let fix a finite set $V \subset [0, 1]$ containing 0 and closed under $\sim$. As a matter of fact, the approach chosen in this section cannot be easily generalised to the infinite case; it has turned out that the use of an infinite set like the rational unit interval would lead to technical difficulties. In the subsequent section, we will modify our approach and the restriction will be dropped.

Definition 3.1. Let S be a nonempty set. A V-valued *fuzzy set* over S is a mapping from S to V.

For two fuzzy sets $u, v\colon S \to V$, we let $u \wedge v$ and $u \vee v$ be the pointwise minimum and maximum of u and v, respectively; we let $\sim u$ the pointwise standard negation of u; and we let $\bar{0}$ and $\bar{1}$ the constant 0 and 1 fuzzy set, respectively. Let M be a collection of V-valued fuzzy sets over S containing $\bar{0}, \bar{1}$ and closed under $\wedge, \vee, \sim$; then we call $(M; \wedge, \vee, \sim, \bar{0}, \bar{1})$ a *Kleene algebra of fuzzy sets*.

For any $u \in M$ and $t \in V$, we define

$$[u]_t \;=\; \{a \in S\colon u(a) = t\}.$$

The Boolean algebra of subsets of S generated by $[u]_t$, where $u \in M$ and $t \in V$, will be called the *Boolean algebra associated with* M, denoted by $\mathcal{B}_M$.

Finally, let ϱ be a rejection function on $\mathcal{B}_M$ w.r.t. $\wedge$. Then (M, ϱ) is called a *Kleene uncertainty algebra*.

We will now define the Possibilistic Logic with Sharp Gradation, denoted by $\mathtt{IG}^0$.

Definition 3.2. The *gradable propositions* of $\mathtt{IG}^0$ are built up from *variables* $\varphi_1, \ldots, \varphi_N$ and the constants $0, 1$ by means of the binary connectives $\wedge, \vee$ and the unary connective $\sim$. We denote the set of gradable propositions by $\mathcal{F}$. The *graded propositions* of $\mathtt{IG}^0$ are of the form (φ, t) where φ is a gradable proposition and $t \in V$.

The *crisp propositions* of $\mathtt{IG}^0$, or *propositions* for short, are built up from graded propositions and the *truth constants* $\bot, \top$ by means of the binary connectives $\wedge, \vee$ and the unary connective $\neg$. We denote the set of propositions by $\mathcal{P}$.

An *implication* of $\mathtt{IG}^0$ is a triple consisting of a finite non-empty set of propositions $\alpha_1, ..., \alpha_k$, a proposition β, and an element $d \in V$, denoted $\alpha_1, ..., \alpha_k \stackrel{d}{\Rightarrow} \beta$.

An *evaluation* for $\mathtt{IG}^0$ is, for some Kleene uncertainty algebra (M, ϱ), a pair of mappings $v_f \colon \mathcal{F} \to M$ and $v_b \colon \mathcal{P} \to M_B$ such that the following holds:

(i) $v_f(\varphi \wedge \psi) = v_f(\varphi) \wedge v_f(\psi)$, $v_f(\varphi \vee \psi) = v_f(\varphi) \vee v_f(\psi)$, $v_f(\sim\varphi) = \sim v_f(\varphi)$ for gradable propositions φ, ψ, and $v_f(0) = \bar{0}$, $v_f(1) = \bar{1}$;

(ii) for $\varphi \in \mathcal{F}$ and $t \in V$ we have

$$v_b((\varphi, t)) \;=\; [v_f(\varphi)]_t \tag{3}$$

and furthermore $v_b(\alpha \wedge \beta) = v_b(\alpha) \wedge v_b(\beta)$, $v_b(\alpha \vee \beta) = v_b(\alpha) \vee v_b(\beta)$, $v_b(\alpha \wedge \beta) = \neg v_b(\alpha)$ for propositions α, β, and $v_b(\bot) = 0$, $v_b(\top) = 1$.

The notions of *satisfaction*, of a *theory*, and of *semantic entailment* is defined for $\mathtt{IG}^0$ similarly as for $\mathtt{I}$.

Note that the variables are now gradable propositions and in fact are interpreted by fuzzy sets. If this is not intended for some variable φ, we may make use of the fact that the implications $(\varphi, 0) \stackrel{1}{\Rightarrow} \neg(\varphi, 1)$ and $(\varphi, 1) \stackrel{1}{\Rightarrow} \neg(\varphi, 0)$ are satisfied only if φ is interpreted by a characteristic function; they can be asserted to ensure crispness. Moreover, if a variable φ is not going to be connected with further gradable variables, this is not even necessary; simply $(\varphi, 1)$ can be used to model a crisp property.

We axiomatise the logic $\mathtt{IG}^0$ as follows.

Definition 3.3. The rules of $\mathtt{IG}^0$ split into three groups:

The *basic rules* are those of $\mathtt{I}$ (see Def. 2.3) where propositions are understood as those of $\mathtt{IG}^0$.

The *degree-of-presence rules* are the following, where φ is a gradable proposition, and $s, s_0, ..., t \in V$:

$$(\varphi, s) \stackrel{d}{\Rightarrow} \neg(\varphi, t) \text{ where } s \neq t$$

$$\neg(\varphi, s_0), \ldots, \neg(\varphi, s_M) \stackrel{d}{\Rightarrow} \bot \text{ where } V = \{s_0, \ldots, s_M\}$$

The *fuzzy-set rules* are the following, where Γ is a finite set of propositions, φ, ψ are gradable propositions, α is a proposition, $c, d \in [0,1]$, and $s, t \in V$:

$$\frac{\Gamma, (\varphi \wedge \psi, s{\wedge}t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s), (\psi, t) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, r), (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } r, s > t$$

$$\frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s < t \qquad \frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s < t$$

$$\frac{\Gamma, (\varphi \vee \psi, s{\vee}t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s), (\psi, t) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, r), (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } r, s < t$$

$$\frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s > t \qquad \frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s > t$$

$$\frac{\Gamma, (\varphi, c) \overset{d}{\Rightarrow} \alpha}{\Gamma, ({\sim}\varphi, {\sim}c) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, ({\sim}\varphi, c) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, {\sim}c) \overset{d}{\Rightarrow} \alpha}$$

The notion of a *proof* of some implication from a theory as well as the *consistency* of a theory is defined like for `I`.

The soundness causes again no difficulties.

Theorem 3.4. *Let $\mathcal{T}$ be a theory of* $\mathtt{IG}^0$ *and $\Gamma \overset{e}{\Rightarrow} \delta$ an implication of* $\mathtt{IG}^0$*. Then $\mathcal{T}$ semantically entails $\Gamma \overset{e}{\Rightarrow} \delta$ if $\mathcal{T}$ proves $\Gamma \overset{e}{\Rightarrow} \delta$.*

For the completeness proof, several preparatory lemmas are necessary. Our procedure in case of the logic $\mathtt{I}^{\odot}$ will serve as a pattern.

In what follows, by a graded variable we will mean a graded proposition (φ, t) such that φ is a variable. In our first step we will show that compound gradual propositions are eliminable from the calculus; graded propositions are replacable by Boolean combination of graded variables.

Lemma 3.5. *Let φ, ψ be gradable propositions of* $\mathtt{IG}^0$*, and let $t \in V$. Then $(\varphi \wedge \psi, t)$ is equivalent to*

$$\bigvee\{(\varphi, r) \wedge (\psi, s) \colon\; r = t \text{ and } s \geq t, \;\; \text{or } r \geq t \text{ and } s = t\}; \tag{4}$$

$(\varphi \vee \psi, t)$ is equivalent to

$$\bigvee\{(\varphi, r) \wedge (\psi, s) \colon\; r = t \text{ and } s \leq t, \;\; \text{or } r \leq t \text{ and } s = t\}, \tag{5}$$

and $(\sim\varphi\,,t)$ *is equivalent to*

$$(\varphi, \sim t). \tag{6}$$

Furthermore, $\neg(\varphi,t)$ *is equivalent to* $\bigvee_{s\neq t}(\varphi,s)$. *Finally, any proposition of* $\mathtt{IG}^\zeta$ *is equivalent to the disjunction of conjunctions of graded variables.*

Proof. Every disjunct in (4), so (4) itself, implies $(\varphi\wedge\psi,t)$. Furthermore, the negation of (4) is equivalent to the disjunction of $\bigvee_{r,s>t}((\varphi,r)\wedge(\psi,s))$ and $\bigvee_{r<t}(\varphi,r)$ and $\bigvee_{s<t}(\psi,s)$, each of which implies $\neg(\varphi\wedge\psi,t)$.

Similarly, we proceed for (5). The claims concerning $(\sim\varphi\,,t)$ and $\neg(\varphi,t)$ are easy.

The last assertion follows, for a graded proposition, by induction over the complexity of the involved gradable proposition. For a proposition, the assertion follows by induction over its complexity. □

We recall that PV is the Boolean algebra of subsets of V, where V is the set of degrees of presence. Furthermore, we denote by PV^N the N-fold free product of the Boolean algebras PV; for the notion of free products of algebras see, e.g., [Gra, Chapter VI]. We may, and we actually will, identify PV^N with $P(V^N)$, the Boolean algebra of subsets of V^N.

Lemma 3.6. *For propositions* α *and* β *of* $\mathtt{IG}^0$, *we put* $\alpha\approx\beta$ *if* $\mathtt{IG}^0$ *proves* $\alpha \overset{1}{\Rightarrow} \beta$ *and* $\beta \overset{1}{\Rightarrow} \alpha$. *Then the quotient* $\langle\mathcal{P}\rangle$ *of* $\mathcal{P}$ *w.r.t.* $\approx$, *endowed with the induced operations* $\wedge,\vee,\neg$ *and the constants* $\langle\bot\rangle,\langle\top\rangle$, *is a Boolean algebra isomorphic to* PV^N. *The isomorphism is given by*

$$w(\langle(\varphi_i,t)\rangle) \;=\; \{(r_1,...,r_N)\in V^N\colon\, r_i=t\},\quad i=1,\dots,N,\;\; t\in V. \tag{7}$$

Furthermore, let

$$u_i\colon\, V^N\to V,\;\;(r_1,...,r_N)\mapsto r_i,$$

and let $\bar{M}$ *be the Kleene algebra generated by* $u_1,\dots,u_N$. *Then* $\mathcal{B}_{\bar{M}} = PV^N$. *Define* $\varrho\colon\,\mathcal{B}_{\bar{M}}\to[0,1]$ *to be* 0 *on all non-zero elements; then* $(\bar{M},\varrho)$ *is a Kleene uncertainty algebra.*

Define $\bar{v}_f(\varphi_i)=u_i$ *for* $i=1,\dots,N$, *and extend* $\bar{v}_f$ *to* $\mathcal{F}$ *such that* $\wedge,\vee,\sim$ *and* $\bar{0},\bar{1}$ *are preserved. Define* $\bar{v}_b(\alpha)=w(\langle\alpha\rangle)$ *for* $\alpha\in\mathcal{P}$. *Then* $(\bar{v}_f,\bar{v}_b)$ *is an evaluation of* $\mathtt{IG}^0$ *such that* $\bar{v}_b(\alpha)=\emptyset$ *if and only if* $\vdash\alpha\overset{1}{\Rightarrow}\bot$.

Proof. Note first that in all the degree-of-presence and fuzzy-set rules, we may w.l.o.g. assume that $d=1$. Let us modify $\mathtt{IG}^0$ as follows: We drop all fuzzy-set rules and add as new axioms the six implications expressing the equivalences of

$(\varphi \wedge \psi, t)$, $(\varphi \vee \psi, t)$, $(\sim \varphi, t)$ with the expressions (4), (5), and (6), respectively, where φ, ψ are gradable propositions and $t \in [-\zeta, 1+\zeta]$. By Lemma 3.5 all these implications are provable, and from the added axioms we may easily derive any of the dropped rules. So the change has no effect for the set of provable implications.

Note next that a proof of an implication of the form $\alpha \stackrel{1}{\Rightarrow} \beta$ in $\mathtt{IG}^0$ can be chosen such that all occurring degrees of certainty are equal to 1. Let $\mathtt{IG}^0_{\mathrm{c}}$ be the calculus differing from $\mathtt{IG}^0$ in that only degree of certainty 1 are allowed. In $\mathtt{IG}^0_{\mathrm{c}}$, the relation $\approx$ obviously does not change.

By Lemma 2.4, $\mathtt{IG}^0_{\mathrm{c}}$ can be viewed as an extension of classical propositional logic: the variables are identified with the graded propositions; and the extension consists of the axioms of $\mathtt{IG}^0_{\mathrm{c}}$ where each implication $\alpha \stackrel{1}{\Rightarrow} \beta$ is understood as $\alpha \rightarrow \beta$. We keep this viewpoint implicitly in the background. We get as an immediate consequence that $\langle \mathcal{P} \rangle$ is a Boolean algebra.

Each graded proposition in which a compound gradual proposition occurs, is by assumption equivalent to an expression in graded variables. So to determine the Boolean algebra $\langle \mathcal{P} \rangle$, we need to consider only $\langle (\varphi_i, t) \rangle$ where $i \in \{1, ..., N\}$ and $t \in V$.

Consider now the degree-of-presence rules. We can restrict them to the case of graded variables. Indeed, it is not difficult, based on an inductive argument, to derive these axioms for compound gradable propositions from those for graded variables.

So we are left with the degree-of-presence rules restricted to graded variables. These axioms split into N disjoint subsets, one for each i. Furthermore, for any $i \in \{1, ..., N\}$, the Boolean subalgebra $\langle \mathcal{P} \rangle_i$ of $\langle \mathcal{P} \rangle$ generated by $\langle (\varphi_i, t) \rangle$, $t \in V$, is clearly isomorphic to PV under the assignment

$$w_i \colon \langle \mathcal{P} \rangle_i \to PV, \;\; \langle (\varphi_i, t) \rangle \mapsto \{t\}.$$

Consequently, $\langle \mathcal{P} \rangle$ itself is isomorphic to the free product of N copies of PV under that assignment (7). The proof of the first half of the theorem is complete.

Clearly, $(\bar{M}, \varrho)$ is a Kleene uncertainty algebra such that $\mathcal{B}_{\bar{M}} = PV^N$.

It is furthermore clear that $\bar{v}_b$ preserves the Boolean structure of $\mathcal{P}$, and like in the proof of Theorem 2.7 we see that $\bar{v}_b(\alpha) = \emptyset$ iff $\vdash \alpha \stackrel{1}{\Rightarrow} \bot$. Moreover, $\bar{v}_f$ preserves the Kleene structure of $\mathcal{F}$ by construction.

To establish that $(\bar{v}_f, \bar{v}_b)$ is an evaluation, it remains to check (3), that is, we have to show $w(\langle (\varphi, t) \rangle) = [\bar{v}_f(\varphi)]_t$ for all $\varphi \in \mathcal{F}$ and $t \in V$. If φ is a variable, this equation holds by construction. For the general case, we proceed by induction over the complexity of φ and use Lemma 3.5. $\square$

We next note that lemma 2.6 holds a fortiori also for $\mathtt{IG}^0$.

Theorem 3.7. *Let $\mathcal{T}$ be a consistent finite theory of $\mathtt{IG}^0$ and $\Gamma \stackrel{e}{\Rightarrow} \delta$ an implication of $\mathtt{IG}^0$. If $\mathcal{T}$ semantically entails $\Gamma \stackrel{e}{\Rightarrow} \delta$, then $\mathcal{T}$ proves $\Gamma \stackrel{e}{\Rightarrow} \delta$.*

Proof. It is easily checked that all rules are sound. The "if" part follows.

We can assume that $\mathcal{T} = \{\chi_0 \stackrel{d_0}{\Rightarrow} \bot,\ \chi_1 \stackrel{d_1}{\Rightarrow} \bot,\ \ldots,\ \chi_m \stackrel{d_m}{\Rightarrow} \bot\}$, where the χ_i are pairwise disjoint and jointly exhaustive, and $1 = d_0 > \ldots > d_m = 0$.

Let $(\bar{v}_f, \bar{v}_b)$ be the evaluation in $(\bar{M}, \varrho)$ according to Lemma 3.6. Let $S = \bar{v}_b(\neg\chi_0)$, and let M be the Kleene algebra generated by $u_i|_S$, $i = 1, \ldots, N$. Then $\mathcal{B}_M = PS$.

Let $v_f \colon \mathcal{F} \to M,\ \ \varphi \mapsto \bar{v}_f(\varphi)|_S$ and $v_b \colon \mathcal{P} \to \mathcal{B}_S,\ \ \alpha \mapsto \bar{v}_b(\alpha) \cap S$.

We define

$$\varrho \colon \mathcal{B}_M \to [0,1], \quad A \mapsto \min\{d_i \colon 1 \le i \le m \text{ and } A \cap v(\chi_i) \ne \emptyset\},$$

where the minimum of the empty set is 1. This is obviously a rejection function.

Then (v_f, v_b) is an evaluation in the Kleene uncertainty algebra (M, ϱ). Since $\varrho(\chi_i) = d_i$ for all i, all elements of $\mathcal{T}$ are satisfied by (v_f, v_b).

If $\mathcal{T}$ does not prove $\Gamma \stackrel{e}{\Rightarrow} \delta$, we conclude like in the proof of Theorem 2.7 that $\Gamma \stackrel{e}{\Rightarrow} \delta$ is not satisfied by v. This completes the proof of the "only if" part. $\square$

We again note that the completeness theorem could obviously be modified so as to involve finite Kleene uncertainty algebras only.

4 Inclusion of graded properties: the continuous case

A property φ is called vague if not under all circumstances it can be told if φ applies or not. We have proposed to model this generalised type of a property in the usual way: as a fuzzy set over the set of all considered situations. A vague property φ is furthermore characterised by a continuous transition from φ to non-φ. Hence it would actually make sense to allow φ to be assigned any degree of presence taken from the real unit interval $[0,1]$, rather than using a finite subset of $[0,1]$ as we did in the previous section.

The statements (φ, t), where t varies over $[0, 1]$, let us then distinguish between an infinity of pairwise exclusive situations. This fact in turn is not well in line with the idea that (φ, t) reflects an agent's impression, given the fact that there is no infinity of situations observable as pairwise exclusive. Nevertheless our intention might be to work with a continuity of situations. What we propose in this case is to assume that close situations are not necessarily distinguishable. We may wonder what it actually means that an agent is asked to evaluate φ and answers 0.3. In fact, such an answer might mean not more than that φ is neither true nor false but fits somewhat better to the latter possibility, and this explanation fits equally well, say, to the value 0.28 or 0.32.

Our second proposal, which is the topic of the present section, is to postulate that graded propositions (φ, s) and (φ, t) are treated as mutually exclusive only if s and t differ at least by a fixed minimal value, denoted by ζ. We will modify the interpretation of (φ, t), $t \in [0, 1]$ accordingly. If φ is interpreted by a fuzzy set u over a set S, we have interpreted (φ, t) by now by $[u]_t$; in what follows we will use $[u]_t^\zeta$ instead, which is, roughly speaking, the set of those elements of S which map to a degree differing from t less than ζ.

We quickly add that this idea is unrelated to any of the formalisms based on interval-valued fuzzy sets. In fact, gradable propositions will still be modelled by ordinary fuzzy sets. What we intend to account for is rather the idea that statements involving truth degrees should have a more "tolerant" interpretation; close truth degrees are allowed to overlap in their interpretation.

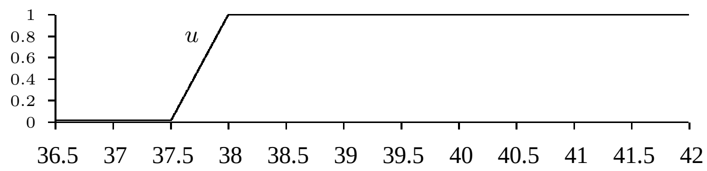
There is one technical aspect which we have to take into account. To replace $[u]_t$ by the larger set $[u]_t^\zeta$ makes perfect sense if t is an intermediate truth degree, in particular if $\zeta \leq t \leq 1 - \zeta$. For sharp truth degrees, the situation is different; we should still be able to refer to the sets $[u]_0$ and $[u]_1$. For, not to be able to tell that a property is false but only being able to say that the degree of presence is below ζ, is an inacceptable restriction. For this reason we will extend the set of available degrees of presence from $[0, 1]$ to $[-\zeta, 1 + \zeta]$. The negative degrees and the degree larger than 1 are so-to-say virtual ones. A degree $c \in (-\zeta, 0)$ represents falsity, like 0, but in contrast to 0 the tolerance around 0 can be arbitrarily small. The degree $-\zeta$ represents clear falsity. Similarly, we use the degrees of presence above 1.

Remark 4.1. *Our formalism could be simplified in an easy way: we could interpret (φ, t) by the set of all $a \in S$ which are mapped to t or a larger value. This is indeed an interpretation common in fuzzy logics. It would include, up to marginal points, the concept of intervals used in this section sand the concept of points from the previous section. Even better, the fuzzy-set rules would simplify and in fact look* *more elegant. However, we do not adopt this approach here. We would see it as a lack of elegancy to declare statements like "property φ holds to a degree of at least* 0.4*" as basic. An agent's utterance of this form, or a technical specification in this form, would come as a surprise.*

We admit that statements providing a lower bound for the degree of presence of some property may reasonably occur as the result of some inference step. Still, we believe that the deeper reason for the common interpretation of syntactically provided degrees as lower bounds in fuzzy logics is of a formal nature. In particular, in fuzzy logics with evaluated syntax [NPM, Haj] *the modus ponens would not be sound if lower boundedness was replaced by equality.*

As a further consequence of our decision to work with the sets $[u]_t^\zeta$ which involve an "extended" set of degrees around t, we will no longer use the set-theoretical operations to interpret Boolean connectives. Our motivation is that single degrees of presence should no longer play a role. It should not matter if the marginal points $t-\zeta$ or $t+\zeta$ are included or not – unless they equal 0 or 1. Furthermore, by use of Boolean connectives it should never be possible to arrive at sets of the form $[u]_t$, $0 < t < 1$.

Hence we need to endow our fuzzy set model with more structure than before. As a prototype consider the following simple fuzzy set, modelling "having fever"; the base set is the set of possible body temperatures in $^\circ C$:



We abstract from this example the following facts. The base set and the set of degrees of presence are endowed with a topology in a natural way, and w.r.t. these topologies the fuzzy set u is continuous. Furthermore, both regions where the modelled property has a clear truth degree are extended and a specific intermediate degree is assigned for a single point only. Topologically speaking, we observe that each set $[u]_t$, where $0 < t < 1$, has an empty interior; that each set $\{s \in [36.5, 42]\colon\ t - \zeta < u(s) < t + \zeta\}$ is the interior of a closed set; and that $[u]_0$ and $[u]_1$ are the closure of open sets. This is the background on which our subsequent considerations are based; the notion of a regular open set will be central.

Remark 4.2. We compile for what follows the basics about the used topological notions. For more information see e.g. [GiHa].

Let S be a topological space. For $A \subseteq S$, we denote by A° the open interior of A, and by A^- the closure of A. A set $A \subseteq S$ is called regular open if it is the open interior of a closed set. So exactly all sets of the form $A^{-\circ}$ are regular open; we have

$$A^{-\circ} \;=\; \{x \in S\colon\ A \text{ is dense in some open neighborhood of } x\}.$$

We denote by $\mathcal{R}(S)$ the set of all regular open subsets of S. Under set-theoretical inclusion, $\mathcal{R}(S)$ is a distributive $0,1$-lattice. For $A, B \in \mathcal{R}(S)$, the infimum is $A \cap B$; the supremum is $A \vee B = (A \cup B)^{-\circ}$; and $\emptyset, S$ are the bottom and top element, respectively. Furthermore,

$$^\perp\colon\ \mathcal{R}(S) \to \mathcal{R}(S), \quad A \mapsto (S \backslash A)^\circ$$

is a complementation function; in particular, $A^{\perp\perp} = A^{-\circ} = A,\ A \cap A^\perp = \emptyset$, and $A \cup A^\perp$ is dense in S. So $(\mathcal{R}(S); \cap, \vee, ^\perp, \emptyset, S)$ is a Boolean algebra.

For later use we remark the following. For open sets $A, B \subseteq S$ we have

$$(A \cup B)^{-\circ} \;=\; A^{-\circ} \vee B^{-\circ}, \tag{8}$$

$$(A \cap B)^{-\circ} \;=\; A^{-\circ} \cap B^{-\circ}, \tag{9}$$

where the supremum refers to the poset $\mathcal{R}(S)$.

Note first that for any $C, D \subseteq S$ we have $(C \cup D)^\perp = C^\perp \cap D^\perp$ and $C^{\perp\perp} = C^{-\circ}$. We conclude

$$\begin{aligned}(A \cup B)^{-\circ} =&(A \cup B)^{\perp\perp} = (A^\perp \cap B^\perp)^\perp = (A^{\perp\perp\perp} \cap B^{\perp\perp\perp})^\perp = \\ &(A^{\perp\perp} \cup B^{\perp\perp})^{\perp\perp} = (A^{-\circ} \cup B^{-\circ})^{-\circ} = A^{-\circ} \vee B^{-\circ}.\end{aligned}$$

This is (8); for (9) see [GiHa, Lemma 4 of Chapter 10].

Definition 4.3. Let S be a topological space. A fuzzy set $u\colon\ S \to [0, 1]$ is called *regular* if the following conditions hold:

(R1) u is continuous w.r.t. the standard topology of $[0, 1]$;

(R2) for any $t \in (0, 1)$, $[u]_t$ has an empty interior.

A Kleene algebra M of regular fuzzy sets over S is called *regular*.

The notion of a regular Kleene algebra would not make sense if the conditions (R1) and (R2) were not preserved under the Kleene algebra operations.

Lemma 4.4. *Let M be a Kleene algebra of fuzzy sets over some topological space. Assume that M is generated by regular fuzzy sets. Then M is a regular Kleene algebra.*

Proof. Evidently, the constant fuzzy sets are regular. Let $u, v \in M$ be regular. Clearly, $u \wedge v$, $u \vee v$, and $\sim u$ are continuous.

Let $0 < t < 1$; we have to show that $[u \wedge v]_t$ has an empty interior. Let $a \in S$ be such that $(u \wedge v)(a) = t$, and let U be an open neighborhood of a. W.l.o.g. assume $t = u(a) \leq v(a)$. If there is a $b \in U$ such that $u(b) < t$, we have $(u \wedge v)(b) = u(b) \wedge v(b) < t$. Otherwise there is a $b \in U$ such that $u(b) > t$ and consequently there is an open $V \subseteq U$ such that $u(b) > t$ for all $b \in V$. Choose some $c \in V$ such that $v(c) \neq t$; then $(u \wedge v)(b) \neq t$. It follows that $[u \wedge v]_t$ does not contain an open set.

Similarly we argue in case of $[u \vee v]_t$. Finally $[\sim u]_t = [u]_{\sim t}$ clearly has an empty interior as well. □

We fix now a rational value $0 < \zeta < \frac{1}{2}$. ζ is supposed to quantify the distinguishability between different degrees to which a vague property holds; (φ, s) and (φ, t) will be modelled as disjoint only if $|s - t| \geq 2\zeta$. We will switch from $[u]_t$ to $[u]_t^\zeta$; here $[u]_t^\zeta$ does not simply denotes the set of all point mapping to the interval $[t-\zeta, t+\zeta]$ or $(t-\zeta, t+\zeta)$; we will rather use a definition which ensures that $[u]_t^\zeta$ is regular open. Accordingly, rather than using the Boolean algebra generated by the sets $[u]_t$, we will work with the Boolean algebra of regular open sets generated by the sets of the form $[u]_t^\zeta$.

For a generalised degree of presence t, t' will denote the degree of presence which is actually meant, disregarding the amount of tolerance: for $t \in [-\zeta, 1+\zeta]$, we put $t' = (t \vee 0) \wedge 1$. For $I \subseteq [-\zeta, 1+\zeta]$, we will write $I' = \{t' \colon t \in I\}$.

Finally, for $I \subseteq [0, 1]$, we put $[u]_I = \{a \in S \colon u(a) \in I\}$.

Definition 4.5. Let M be a regular Kleene algebra of fuzzy sets over a topological space S. For $u \in M$ and $t \in [-\zeta, 1+\zeta]$, we define

$$[u]_t^\zeta \;=\; [u]_{(t-\zeta, t+\zeta)'}{}^{-\circ}.$$

Furthermore, the Boolean subalgebra of $\mathcal{R}(S)$ generated by $[u]_t^\zeta$, where $u \in M$ and $t \in [-\zeta, 1+\zeta]$, will be called the *Boolean algebra associated with M*, denoted by $\mathcal{R}_M$.

Finally, let ϱ be a rejection function on $\mathcal{R}_M$ w.r.t. $\wedge$. Then (M, ϱ) is called a *regular Kleene uncertainty algebra*.

The following lemma provides an explicit description of the interpretation which we are going to apply.

Lemma 4.6. *Let (M, ϱ) be a regular Kleene uncertainty algebra. Let $u \in M$. For $\zeta \leq t \leq 1 - \zeta$ we have*

$$\begin{aligned} [u]_t^\zeta &= {[u]_{(t-\zeta,t+\zeta)}}^{-\circ} \\ &= \{a \in S\colon\ u(a) \in (t-\zeta, t+\zeta) \\ &\qquad \text{or } u(a) \in \{t-\zeta, t+\zeta\} \text{ and } [u]_{(t-\zeta,t+\zeta)} \text{ is dense in some neighborhood of } a\}; \end{aligned}$$

for $-\zeta < t \leq \zeta$, we have

$$\begin{aligned} [u]_t^\zeta &= {[u]_{[0,t+\zeta)}}^{-\circ} \\ &= \{a \in S\colon\ u(x) \in [0, t+\zeta) \\ &\qquad \text{or } u(a) = t+\zeta \text{ and } [u]_{[0,t+\zeta)} \text{ is dense in some neighborhood of } a\}; \end{aligned}$$

and similarly for $1 - \zeta \leq t < 1 + \zeta$. Finally,

$$\begin{aligned} [u]_{-\zeta}^\zeta &= {[u]_0}^\circ, \\ [u]_{1+\zeta}^\zeta &= {[u]_1}^\circ. \end{aligned}$$

So given a regular fuzzy set u, we see that $[u]_t^\zeta$ contains basically all points $a \in S$ such that $u(a) \in (t-\zeta, t+\zeta)$, but if, for instance, u has at the point $a \in S$ the strict local minimum $t - \zeta$ then a is joined to $[u]_t^\zeta$ as well. Furthermore, the property associated to u to be clearly false is modelled by the set $[u]_{-\zeta}^\zeta$; this set is contained in $[u]_0$ and contains an $a \in S$ only if it is in the interior of the set of the elements mapped to 0.

In the above example, take $\zeta = 0.1$. Then we have, say, $[u]_{0.3}^\zeta = u^{-1}((0.2, 0.4)) = (37.6, 37.7)$, $[u]_{0.1}^\zeta = u^{-1}((0, 0.2)) = (37.5, 37.6)$. The associated crisp properties are $[u]_{-0.1}^\zeta = [36.5, 37.5)$ and $[u]_{1.1}^\zeta = (38, 42]$.

We define the Possibilistic Logic with Soft Gradation, denoted by $\mathtt{IG}^\zeta$, as follows.

Definition 4.7. The *propositions*, the set of which will still be denoted by $\mathcal{P}$, as well as the *implications* of $\mathtt{IG}^\zeta$ coincide with those of $\mathtt{IG}^0$, respectively (see Def. 3.2) except that we use the real interval $[-\zeta, 1 + \zeta]$ as the set of degrees of presence.

An *evaluation* v of $\mathtt{IG}^\zeta$ is defined like for $\mathtt{IG}^0$ except that for $\varphi \in \mathcal{F}$ and $t \in [-\zeta, 1 + \zeta]$ we define

$$v_b((\varphi, t)) \;=\; [v_f(\varphi)]_t^\zeta$$

and that v_b maps to $\mathcal{R}_M$.

A *theory* of $\mathtt{IG}^\zeta$ and *semantic entailment* for $\mathtt{IG}^\zeta$ is defined mutatis mutandis like for $\mathtt{I}$.

To axiomatise the logic $\mathtt{IG}^\zeta$ we have to modify all rules except the basic ones.

Definition 4.8. The rules of $\mathtt{IG}^\zeta$ split into three groups:

The *basic rules* are those of $\mathtt{I}$ (see Def. 2.3) where propositions are understood as those of $\mathtt{IG}^\zeta$.

The *degree-of-presence rules* are the following, where φ is a gradable proposition and $s, s_1, ..., t \in [-\zeta, 1+\zeta]$:

$$(\varphi, s) \overset{d}{\Rightarrow} \neg(\varphi, t) \text{ where } |s-t| \geq 2\zeta$$

$$(\varphi, s) \overset{d}{\Rightarrow} (\varphi, t) \text{ where } -\zeta \leq s \leq t < \zeta \text{ or } 1-\zeta < t \leq s \leq 1+\zeta$$

$$(\varphi, r) \overset{d}{\Rightarrow} (\varphi, s) \vee (\varphi, t) \text{ where } s \leq r \leq t \leq s+2\zeta$$

$$(\varphi, r), (\varphi, s) \overset{d}{\Rightarrow} (\varphi, t) \text{ where } r \leq t \leq s$$

$$\neg(\varphi, s_1), \ldots, \neg(\varphi, s_k) \overset{d}{\Rightarrow} \bot \text{ where } s_1 < \zeta;\ s_2 - s_1, \ldots, s_k - s_{k-1} \leq 2\zeta;\ s_k > 1-\zeta$$

The *fuzzy-set rules* are the following, where φ, ψ are gradable propositions, α is a proposition, Γ is a finite set of propositions, $c, d \in [0,1]$, and $s, t \in [-\zeta, 1+\zeta]$:

$$\frac{\Gamma, (\varphi \wedge \psi, s \curlywedge t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s), (\psi, t) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, r), (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } r, s \geq t+2\zeta$$

$$\frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s+2\zeta \leq t \qquad \frac{\Gamma, \neg(\varphi \wedge \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s+2\zeta \leq t$$

$$\frac{\Gamma, (\varphi \vee \psi, s \curlyvee t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s), (\psi, t) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, r), (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } r+2\zeta, s+2\zeta \leq t$$

$$\frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s \geq t+2\zeta \qquad \frac{\Gamma, \neg(\varphi \vee \psi, t) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\psi, s) \overset{d}{\Rightarrow} \alpha} \text{ where } s \geq t+2\zeta$$

$$\frac{\Gamma, (\varphi, c) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\sim\varphi, \sim c) \overset{d}{\Rightarrow} \alpha} \qquad \frac{\Gamma, (\sim\varphi, c) \overset{d}{\Rightarrow} \alpha}{\Gamma, (\varphi, \sim c) \overset{d}{\Rightarrow} \alpha}$$

The notion of a *proof* as well as the *consistency* of a theory is defined like for $\mathtt{I}$ (see Def. 2.3).

We split up the soundness and completeness proof for $\mathtt{IG}^\zeta$ in a series of lemmas.

To establish the soundness of the rules, we have to examine the structure of our model in some more detail. In the next lemma we see how the Boolean operations act, for some fixed fuzzy set u, on the sets $[u]^\zeta_t$ where $t \in [-\zeta, 1+\zeta]$.

Lemma 4.9. *Let (M, ϱ) be a regular Kleene uncertainty algebra. Let $u \in M$, and let $\mathcal{R}_u$ be the Boolean subalgebra generated by $[u]^\zeta_t$, $t \in [-\zeta, 1+\zeta]$ in $\mathcal{R}_M$.*

Let $[-\frac{1}{2}, 1\frac{1}{2}]$ be endowed with the standard topology, and let $\mathcal{R}([-\frac{1}{2}, 1\frac{1}{2}])$ be the Boolean algebra of regular open subsets of $[-\frac{1}{2}, 1\frac{1}{2}]$. Let furthermore $\mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]}$ be the Boolean subalgebra of $\mathcal{R}([-\frac{1}{2}, 1\frac{1}{2}])$ consisting of $I \in \mathcal{R}([-\frac{1}{2}, 1\frac{1}{2}])$ such that (i) $[-\frac{1}{2}, 0)$ *is fully contained in I or disjoint from I and* (ii) $(1, 1\frac{1}{2}]$ *is fully contained in I or disjoint from I.*

Then the mapping

$$\iota_u \colon \mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]} \to \mathcal{R}_u, \quad I \mapsto {[u]_{I'}}^{-\circ}$$

is an epimorphism of Boolean algebras.

Proof. Note first that $\mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]}$ consists of the finite unions of pairwise not tangent intervals of the form

$$\begin{array}{l} [-\frac{1}{2}, 0), \quad [-\frac{1}{2}, a) \text{ for } 0 < a \leq 1, \quad [-\frac{1}{2}, 1\frac{1}{2}], \\ (a, b) \text{ for } 0 \leq a < b \leq 1, \quad (a, 1\frac{1}{2}] \text{ for } 0 \leq a < 1, \quad (1, 1\frac{1}{2}]. \end{array} \tag{10}$$

Let S be the domain of u. In view of Lemma 4.6, $\mathcal{R}_u$ contains the sets

$$\begin{array}{l} {[u]_0}^\circ, \ {[u]_{[0,s)}}^{-\circ} \text{ for } 0 < s \leq 1, \ S, \\ {[u]_{(s,t)}}^{-\circ} \text{ for } 0 \leq s < t \leq 1, \ {[u]_{(s,1]}}^{-\circ} \text{ for } 0 \leq s < 1, \ {[u]_1}^\circ. \end{array} \tag{11}$$

We see that the sets (11) are the images of the intervals (10) under ι_u, respectively.

For $0 \leq s < t \leq s' < t' \leq 1$, we have by the regularity of u

$${[u]_{(s,t)}}^{-\circ} \vee {[u]_{(s',t')}}^{-\circ} \ = \ \begin{cases} {[u]_{(s,t')}}^{-\circ} & \text{if } t = s', \\ {[u]_{(s,t) \cup (s',t')}}^{-\circ} & \text{if } t < s' \end{cases} \ = \ {[u]_{(s,t) \vee (s',t')}}^{-\circ};$$

by checking similarly all possible combinations of pairs of intervals of the form (10) we see that ι_u preserves $\vee$. In a similar way we see that ι_u preserves $\wedge$ and $\sim$. Clearly, ι_u is surjective. □

For the next lemma we introduce some technical notation. For $t \in [-\zeta, 1+\zeta]$, we define the finite subset $V_{\geq t}$ of $[-\zeta, 1+\zeta]$ to contain all values $t, t+2\zeta, t+4\zeta, \ldots$

which are smaller or equal to $1+\zeta$. Similar, we define $V_{\leq t}$ to contain those values $t, t-2\zeta, \ldots$ which are larger or equal to $-\zeta$. Finally, we let $V_{\neg t}$ contain those values $t-2\zeta, t-4\zeta, \ldots$ as well as $t+2\zeta, t+4\zeta, \ldots$ which are contained in $[-\zeta, 1+\zeta]$. Finally, for an element u of a regular Kleene algebra, we put $[u]^\zeta_{\geq t} = ([u]_{(t-\zeta, 1+\zeta]'})^{-\circ}$ and $[u]^\zeta_{\leq t} = ([u]_{[-\zeta, t+\zeta)'})^{-\circ}$.

Lemma 4.10. *Let (M, ϱ) be a regular Kleene uncertainty algebra. Let $u, v \in M$ and let $t \in [-\zeta, 1+\zeta]$. Then*

$$\begin{aligned}
[u \wedge v]^\zeta_t &= ([u]_t \cap [v]^\zeta_{\geq t}) \vee ([u]^\zeta_{\geq t} \cap [v]_t) \\
&= \bigvee \{[u]^\zeta_r \cap [v]^\zeta_s :\ \ r = t \text{ and } s \in V_{\geq t}, \ \ \text{or } r \in V_{\geq t} \text{ and } s = t\}, \\
[u \wedge v]^\zeta_t &= ([u]_t \cap [v]^\zeta_{\leq t}) \vee ([u]^\zeta_{\leq t} \cap [v]_t) \\
&= \bigvee \{[u]^\zeta_r \cap [v]^\zeta_s :\ \ r = t \text{ and } s \in V_{\leq t}, \ \ \text{or } r \in V_{\leq t} \text{ and } s = t\}, \\
[\sim u]^\zeta_t &= [u]^\zeta_{\sim t}.
\end{aligned}$$

Proof. Using (8) and (9) we calculate

$$\begin{aligned}
[u \wedge v]^\zeta_t &= [u \wedge v]_{(t-\zeta, t+\zeta)'}{}^{-\circ} \\
&= (([u]_{(t-\zeta, t+\zeta)'} \cap [v]_{(t-\zeta, 1+\zeta]'}) \cup ([u]_{(t-\zeta, 1+\zeta]'} \cap [v]_{(t-\zeta, t+\zeta)'}))^{-\circ} \\
&= ([u]_{(t-\zeta, t+\zeta)'} \cap [v]_{(t-\zeta, 1+\zeta]'})^{-\circ} \vee ([u]_{(t-\zeta, 1+\zeta]'} \cap [v]_{(t-\zeta, t+\zeta)'})^{-\circ} \\
&= ([u]_{(t-\zeta, t+\zeta)'}{}^{-\circ} \cap [v]_{(t-\zeta, 1+\zeta]'}{}^{-\circ}) \vee ([u]_{(t-\zeta, 1+\zeta]'}{}^{-\circ} \cap [v]_{(t-\zeta, t+\zeta)'}{}^{-\circ}) \\
&= ([u]^\zeta_t \cap [v]^\zeta_{\geq t}) \vee ([u]^\zeta_{\geq t} \cap [v]^\zeta_t);
\end{aligned}$$

furthermore, we clearly have $[u]^\zeta_{\geq t} = \bigvee_{s \in V_{\geq t}} [u]^\zeta_s$ and $[v]^\zeta_{\geq t} = \bigvee_{s \in V_{\geq t}} [v]^\zeta_s$, and the assertion follows by distributivity. Similarly we proceed for $[u \vee v]^\zeta_t$. The expression for $[\sim u]^\zeta_t$ is obvious. □

Theorem 4.11. *Let $\mathcal{T}$ be a theory of $\mathtt{IG}^\zeta$ and $\Gamma \overset{e}{\Rightarrow} \delta$ an implication of $\mathtt{IG}^\zeta_{\mathcal{T}}$. Then $\mathcal{T}$ semantically entails $\Gamma \overset{e}{\Rightarrow} \delta$ if $\mathcal{T}$ proves $\Gamma \overset{e}{\Rightarrow} \delta$.*

Proof. The basic rules are sound by Theorem 2.7.

The soundness of the degree-of-presence and fuzzy-set rules follows from Lemmas 4.9 and 4.10. □

We will now work towards the completeness part. We will proceed in analogy to the case of the logic $\mathtt{IG}^0$ whenever possible.

We first see how graded propositions decompose to Boolean expressions in graded variables, in the same way as described in Lemma 4.10.

Lemma 4.12. *Let φ, ψ be gradable propositions of $\mathtt{IG}^\zeta$, and let $t \in [-\zeta, 1+\zeta]$. Then $(\varphi \wedge \psi, t)$ is equivalent to*

$$\bigvee \{((\varphi, r) \wedge (\psi, s)) \colon \ r = t \text{ and } s \in V_{\geq t}, \quad \text{or } r \in V_{\geq t} \text{ and } s = t\}; \tag{12}$$

$(\varphi \vee \psi, t)$ is equivalent to

$$\bigvee \{((\varphi, r) \wedge (\psi, s)) \colon \ r = t \text{ and } s \in V_{\leq t}, \quad \text{or } r \in V_{\leq t} \text{ and } s = t\}, \tag{13}$$

and $(\sim \varphi, t)$ is equivalent to $(\varphi, \sim t)$.

Furthermore, $\neg(\varphi, t)$ is equivalent to

$$\bigvee_{s \in V_{\neg t}} (\varphi, s). \tag{14}$$

Finally, any proposition of $\mathtt{IG}^\zeta$ is equivalent to the disjunction of conjunctions of graded variables.

Proof. (12) implies $(\varphi \wedge \psi, t)$. Furthermore, the negation of this proposition is equivalent to a finite disjunction of propositions $(\varphi, r) \wedge (\psi, s)$ where either $r, s \geq t + 2\zeta$ or $r \leq t - 2\zeta$ or $s \leq t - 2\zeta$, each of which implies $\neg(\varphi \wedge \psi, t)$.

Similarly, we proceed to show that $(\varphi \vee \psi), t)$ is equivalent to (13). The assertion about $(\sim \varphi, t)$ is easy.

It is easily seen that $\neg(\varphi, t)$ is equivalent to (14).

By induction over its complexity we conclude that each graded proposition is the disjunction of conjunctions of graded variables. It follows that the same is the case for each proposition. □

For the Boolean algebra $\mathcal{R}(S)$ of open regular sets of a topological space S, we again denote by $\mathcal{R}(S)^N$ the N-fold free product of $\mathcal{R}(S)$. We can, and will, identify $\mathcal{R}(S)^N$ with a subalgebra of $\mathcal{R}(S^N)$, the algebra of regular open sets in the product space S^N; $\mathcal{R}(S)^N$ is generated by the sets of the form $A_1 \times \ldots \times A_N$, where $A_1, \ldots, A_N \in \mathcal{R}(S)$. We will call the sets of the latter form cubic. – An analogous remark applies to any subalgebra of $\mathcal{R}(S)$.

In the following lemma, we define for $t \in [-\zeta, 1+\zeta]$ the set $U_\zeta(t) \in \mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]}$ to be the interval $[-\frac{1}{2}, t+\zeta)$ if $-\zeta \leq t < \zeta$, $(t-\zeta, t+\zeta)$ if $\zeta \leq t \leq 1-\zeta$, and $(t-\zeta, 1\frac{1}{2}]$ if $1-\zeta < t$.

Lemma 4.13. *For propositions α and β of $\mathtt{IG}^\zeta$, we put $\alpha \approx \beta$ if $\mathtt{IG}^\zeta$ proves $\alpha \stackrel{1}{\Rightarrow} \beta$ and $\beta \stackrel{1}{\Rightarrow} \alpha$. Then the quotient $\langle \mathcal{P} \rangle$ of $\mathcal{P}$ w.r.t. $\approx$, endowed with the induced operations $\wedge, \vee, \neg$ and the constants $\langle \bot \rangle, \langle \top \rangle$, is a Boolean algebra isomorphic to ${\mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]}}^N$. The isomorphism w is given by*

$$\begin{aligned} w(\langle(\varphi_i, t)\rangle) \; &= \{(r_1, ..., r_N) \in [-\tfrac{1}{2}, 1\tfrac{1}{2}]^N \colon \; r_i \in U_\zeta(t)\}, \\ & \qquad\qquad\qquad i = 1, \ldots, N, \;\; t \in [-\zeta, 1+\zeta]. \end{aligned} \tag{15}$$

Furthermore, let

$$u_i \colon \; [-\tfrac{1}{2}, 1\tfrac{1}{2}]^N \to [0,1], \quad (r_1, ..., r_N) \mapsto r_i',$$

and let $\bar{M}$ be the Kleene algebra generated by $u_1, \ldots, u_N$. Then $\mathcal{R}_{\bar{M}} = {\mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]}}^N$. Define $\varrho \colon \mathcal{R}_{\bar{M}} \to [0,1]$ to be 0 on all non-zero elements; then $(\bar{M}, \varrho)$ is a regular Kleene uncertainty algebra.

Define $\bar{v}_f(\varphi_i) = u_i$ for $i = 1, \ldots, N$, and extend $\bar{v}_f$ to $\mathcal{F}$ such that $\wedge, \vee, \sim$ and $\bar{0}, \bar{1}$ are preserved. Define $\bar{v}_b(\alpha) = w(\langle \alpha \rangle)$ for $\alpha \in \mathcal{P}$. Then $(\bar{v}_f, \bar{v}_b)$ is an evaluation of $\mathtt{IG}^\zeta$ such that $\bar{v}_b(\alpha) = \emptyset$ if and only if $\vdash \alpha \stackrel{1}{\Rightarrow} \bot$.

Proof. Again, for the degree-of-presence and fuzzy-set rules we may assume $d = 1$. We modify $\mathtt{IG}^\zeta$: We drop the fuzzy-set rules and add six axiom schemes expressing the equivalences of $(\varphi \wedge \psi, t)$, $(\varphi \vee \psi, t)$, $(\sim \varphi, t)$ with the expressions (12), (13), and (14), respectively. By Lemma 4.12 we see that this change has no effect.

Let $\mathtt{IG}^\zeta_\mathtt{c}$ be the restriction of $\mathtt{IG}^\zeta$ to degrees of certainty 1. In the same way as in the proof of Lemma 3.6, we may view $\mathtt{IG}^\zeta_\mathtt{c}$ as an extension of classical propositional logic.

We have to determine the Boolean algebra $\langle \mathcal{P} \rangle$. It is tedious but not difficult to check that the degree-of-presence rules can be restricted to the case of graded variables. Consequently, we again have N disjoint groups of axioms involving for each $i \in \{1, ..., N\}$ the graded variables (φ_i, t), $t \in [-\zeta, 1+\zeta]$.

Fix an $i \in \{1, ..., N\}$. We have to show that the subalgebra $\langle \mathcal{P} \rangle_i$ of $\langle \mathcal{P} \rangle$ generated by $\langle (\varphi_i, t) \rangle$, $t \in [-\zeta, 1+\zeta]$, is isomorphic to $\mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]}$ under the assignment

$$w_i \colon \; \langle \mathcal{P} \rangle_i \to \mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]}, \quad \langle(\varphi_i, t)\rangle \mapsto U_\zeta(t).$$

It will then follow that $\langle \mathcal{P} \rangle$ is isomorphic to the free product of N copies of $\mathcal{R}_{[-\frac{1}{2}, 1\frac{1}{2}]}$, the isomorphism being determined by (15).

It is easily checked that all inequalities holding in $\langle\mathcal{P}\rangle_i$ due to the degree-of-presence rules between $\langle(\varphi_i, t)\rangle$ are preserved by w_i. So w_i indeed extends to a Boolean homomorphism from the whole $\langle\mathcal{P}\rangle_i$ to $\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}$. By construction w_i is surjective.

It remains to show that w_i is injective. Let $\{s_j : j \in J\}$ and $\{t_k : k \in K\}$ be two finite subsets of $[-\zeta, 1+\zeta]$. We have to prove that $\bigcap_j U_\zeta(s_j) \subseteq \bigvee_k U_\zeta(t_k)$ implies that $\mathtt{IG}^\zeta$ proves $\bigwedge_j(\varphi_i, s_j) \overset{1}{\Rightarrow} \bigvee_k(\varphi_i, t_k)$.

Case 1. J is empty. Then $\bigvee_k U_\zeta(t_k) = [-\zeta, 1+\zeta]$ and consequently $\{t_k\}$ contains elements $< \zeta$ and $> 1+\zeta$ and neighboring values differ at most 2ζ. It follows that $\top \overset{1}{\Rightarrow} \bigvee_k(\varphi_i, t_k)$ is provable and the assertion follows.

Case 2. $J = \{j\}$ is one-element. Then either $s_j < \zeta$ and $s_j \le t_k < \zeta$ for some k; or $1-\zeta < s_j$ and $1-\zeta < t_k \le s_j$ for some k; or else there are two values t_k with distance $\le 2\zeta$ and such that s_j is in between. The assertion follows in each case.

Case 3. All ≥ 2 values in $\{s_j\}$ are $< \zeta$ or $> 1-\zeta$. This case reduces to Case 2.

Case 3. Two of the values in $\{s_j\}$ differ at least 2ζ, that is, the intersection $\bigcap_j U_\zeta(s_j)$ is empty. Then $\bigwedge_j(\varphi_i, s_j) \overset{1}{\Rightarrow} \bot$ is provable and the assertion follows.

Case 5. At least one value in $\{s_j\}$ is in $(\zeta, 1-\zeta)$ and all ≥ 2 values have a mutual distance of $\le 2\zeta$. If there are more than two, we can delete all but the outermost ones. Let $s_j, s_{j'}$ be the two values and let $s_j < s_{j'}$. Then either there is a $s_j \le t_k \le s_{j'}$, or there are $t_k < s_j < s_{j'} < t_{k'}$ such that $t_{k'} - t_k < 2\zeta$. The assertion follows in both cases.

The proof of the first half of the theorem is complete. For the second we proceed like in the proof of Lemma 3.6. □

We next note that lemma 2.6 holds a fortiori also for $\mathtt{IG}^\zeta$.

Theorem 4.14. *Let $\mathcal{T}$ be a consistent finite theory of $\mathtt{IG}^\zeta$ and $\Gamma \overset{e}{\Rightarrow} \delta$ an implication of $\mathtt{IG}^\zeta_\mathcal{T}$. If $\mathcal{T}$ semantically entails $\Gamma \overset{e}{\Rightarrow} \delta$, then $\mathcal{T}$ proves $\Gamma \overset{e}{\Rightarrow} \delta$.*

Proof. Assume to the contrary that $\mathcal{T}$ does not prove $\Gamma \overset{e}{\Rightarrow} \delta$. We can assume that $\mathcal{T} = \{\chi_0 \overset{1}{\Rightarrow} \bot,\ \chi_1 \overset{d_1}{\Rightarrow} \bot,\ \ldots,\ \chi_m \overset{d_m}{\Rightarrow} \bot\}$, where the χ_i are pairwise disjoint and jointly exhaustive, and $1 = d_0 > \ldots > d_m = 0$.

Let $(\bar{v}_f, \bar{v}_b)$ be the evaluation in $(\bar{M}, \varrho)$ according to Lemma 4.13. Let $S = \bar{v}_b(\neg\chi_0)$, and let M be the Kleene algebra generated by $u_i|_S$, $i = 1, \ldots, N$. Then M is a regular Kleene algebra. Furthermore $\mathcal{R}_M = \{A \cap S \colon A \in \mathcal{R}_{\bar{M}}\}$.

Let $v_f \colon \mathcal{F} \to M,\ \ \varphi \mapsto \bar{v}_f(\varphi)|_S$ and $v_b \colon \mathcal{P} \to \mathcal{R}_S,\ \ \alpha \mapsto \bar{v}_b(\alpha) \cap S$.

We define

$$\varrho\colon \mathcal{R}_M \to [0,1], \quad A \mapsto \min \{d_i\colon 0 \le i \le m \text{ and } A \cap \bar{v}_b(\chi_i) \ne \emptyset\},$$

where the minimum of the empty set is 1. This is obviously a rejection function on $\mathcal{R}_M$.

Then (v_f, v_b) is an evaluation in the regular Kleene uncertainty algebra (M, ϱ). Since $\varrho(\bar{v}_b(\chi_i)) = d_i$ for all i, all elements of $\mathcal{T}$ are satisfied by (v_f, v_b).

If $\mathcal{T}$ does not prove $\Gamma \overset{e}{\Rightarrow} \delta$, we conclude like in the proof of Theorem 2.7 that $\Gamma \overset{e}{\Rightarrow} \delta$ is not satisfied by v. This completes the proof of the "only if" part. □

5 Smoothing the degree of certainty over gradation

Having started with the (slightly generalised) Possibilistic Logic as our general framework, we have included the possibility to express gradedness of information and we have subsequently modified the interpretation of graded properties. Doing all this, we have not touched the underlying concept of uncertainty; the degree of certainty has remained unrelated to the degrees of presence.

Indeed, in $\mathtt{IG}^{\zeta}$ situations are specified by the propositions $(\varphi_1, t_1), \ldots, (\varphi_N, t_N)$, that is, by the N-tuple $(t_1, \ldots, t_N)$. To each such N-tuple, there is associated the degree of implausibility of the corresponding situation, namely $d = \varrho(v(\varphi_1, t_1) \wedge \ldots \wedge v(\varphi_N, t_N))$ for some interpretation v. We observe that the value d depends on $(t_1, \ldots, t_N)$ completely arbitrarily.

This arbitrariness might not be ideal for practical applications. Similar situations are presumably described by close N-tuples and so the implausibility should depend continuously on the N parameters. In the present chapter, we add a simple rule to our logic with exactly this effect.

We propose the following approach. Situations are specified by N-tuples $(t_1, \ldots, t_N)$; assume that the associated degree of implausibility is d. We add a rule to ensure that a situation characterised by $(s_1, \ldots, s_N)$, where s_i differs from t_i less than λ, is assigned a degree of implausibility of at least $d - \tau\lambda$. In other words, we introduce Lipschitz continuity for ϱ if seen in dependence on the degrees of presence. So for instance, put $\tau = 2$, and assume that we know with certainty d that we can conclude that if property φ fully applies so does ψ, that is, $(\varphi, 1) \overset{d}{\Rightarrow} (\psi, 1)$. In the calculus introduced below we may conclude that $(\varphi, 0.9) \overset{d-0.2}{\Rightarrow} (\psi, 1)$ provided that $d > 0.2$.

Our new rule offers a simple way to prevent ϱ to "jump" when changing continuously from one situation to another one, in that changes are bounded in dependence of the "distance" between situations. This procedure is certainly pragmatic. But it solves in a very direct and transparent way the problem which we have and its effect can be controlled by a deliberate choice of τ.

Let us fix a real parameter $\tau > 0$. Our refined model relies on the natural parametrisation of the Boolean algebra associated to a regular Kleene algebra, defined as follows.

Lemma 5.1. *Let (M, ϱ) be a regular Kleene uncertainty algebra such that M is generated by $u_1, \ldots, u_N$. For $I_1 \times \ldots \times I_N \in {\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}}^N$, where $I_1, \ldots, I_N \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}$, put*

$$\iota(I_1 \times \ldots \times I_N) \;=\; \iota_{u_1} I_1 \cap \ldots \cap \iota_{u_N} I_N,$$

where ι_{u_i}, $i = 1, ..., N$ is given according to Lemma 4.9. Then ι extends to an epimorphism between Boolean algebras from ${\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}}^N$ to $\mathcal{R}_M$.

Proof. For each i, ι_{u_i} is an homomorphism from $\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}$ to $\mathcal{R}_M$ by Lemma 4.9. These homomorphisms are combined to the homomorphism ι from the N-fold free product of $\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}$ to $\mathcal{R}_M$ as indicated [Gra, Chapter VI].

By Lemma 4.10, ι is surjective. ☐

We will now introduce the quasimetric $\mathrm{d}_\infty(\cdot,\cdot)$ on $[-\frac{1}{2}, 1\frac{1}{2}]^N$, which is defined similarly to the supremum metric, but identifies the points in the marginal intervals $[-\frac{1}{2}, 0]$ and $[1, 1\frac{1}{2}]$:

$$\mathrm{d}_\infty((x_1, ..., x_N), (y_1, ..., y_N)) \;=\; \max_i |x'_i - y'_i|, \qquad (x_1, ..., x_N), (y_1, ..., y_N) \in [-\tfrac{1}{2}, 1\tfrac{1}{2}]^N.$$

Occasionally, we will use the ε-neighborhoods w.r.t. this quasimetric; for $p \in [-\frac{1}{2}, 1\frac{1}{2}]^N$ and $\varepsilon > 0$, we put $U_\varepsilon(p) = \{q \in [-\frac{1}{2}, 1\frac{1}{2}]^N \colon \mathrm{d}_\infty(p, q) < \varepsilon\}$.

Definition 5.2. A function $r\colon [-\frac{1}{2}, 1\frac{1}{2}]^N \to [0, 1]$ is called *τ-Lipschitz continuous* if, for any $p, q \in [-\frac{1}{2}, 1\frac{1}{2}]^N$, we have $|r(p) - r(q)| < \tau\lambda$ whenever $\mathrm{d}_\infty(p, q) < \lambda$.

Let (M, ϱ) be a regular Kleene uncertainty algebra such that M is generated by $u_1, \ldots, u_N$; and let $\iota : {\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}}^N \to \mathcal{R}_M$ be defined according to Lemma 5.1. Then ϱ is said to be *induced* by a function $r : D \to [0, 1]$, where D is a dense subset of $[-\frac{1}{2}, 1\frac{1}{2}]^N$, if

$$\varrho(\iota P) \;=\; \inf_{p \in P \cap D} r(p), \qquad P \in {\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}}^N.$$

If in this case r is τ-Lipschitz continuous and the domain of r is the whole $[-\frac{1}{2}, 1\frac{1}{2}]^N$, we say that ϱ is τ*-smooth* w.r.t. $u_1, \ldots, u_N$.

We next extend $\mathrm{d}_\infty(\cdot,\cdot)$ to pairs of subsets in the usual way, both assymmetrical and symmetrical. So $\mathrm{q_H}(\cdot,\cdot)$ is a modified Hausdorff quasimetric on $\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$:

$$\mathrm{q_H}(P,Q) \;=\; \sup_{p\in P}\inf_{q\in Q} \mathrm{d}_\infty(p,q), \qquad P,Q \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N;$$

and $\mathrm{d_H}(\cdot,\cdot)$ is the modified Hausdorff metric on $\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$:

$$\mathrm{d_H}(P,Q) \;=\; \mathrm{q_H}(P,Q) \vee \mathrm{q_H}(Q,P), \qquad P,Q \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N.$$

Note that, for $P_1, P_2, Q_1, Q_2 \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$, we have

$$\mathrm{d_H}(P_1 \vee P_2, Q_1 \vee Q_2) \;\leq\; \mathrm{d_H}(P_1,Q_1) \vee \mathrm{d_H}(P_2,Q_2); \tag{16}$$

furthermore, if P_1, P_2, Q_1, Q_2 are all cubic, we have

$$\mathrm{d_H}(P_1 \cap P_2, Q_1 \cap Q_2) \;\leq\; \mathrm{d_H}(P_1,Q_1) \vee \mathrm{d_H}(P_2,Q_2). \tag{17}$$

Finally, the diameter of some $P \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$ will be meant to be the value $\sup\{\mathrm{d}_\infty(p_1,p_2)\colon\, p_1,p_2 \in P\}$.

Lemma 5.3. *Let* (M,ϱ) *be a regular Kleene uncertainty algebra such that* M *is generated by* $u_1, \ldots, u_N$*; and let* $\iota\colon\; \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N \to \mathcal{R}_M$ *be defined according to Lemma* 5.1*. The following statements are equivalent:*

- (α) ϱ *is* τ*-smooth w.r.t.* $u_1, \ldots, u_N$*.*
- (β) *For any non-empty* $P,Q \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$*,* $|\varrho(\iota P) - \varrho(\iota Q)| < \tau\lambda$ *if* $\mathrm{d_H}(P,Q) < \lambda$*.*
- (γ) *Let* $v_1, \ldots, v_k \in M$ *be expressible from* $u_1, \ldots, u_N$ *such that for each* i*,* u_i *occurs at all places positively or at all places negatively. Furthermore, let* $s_1, t_1, \ldots, s_k, t_k$ *be such that if, for some* i*,* u_i *occurs both in* v_{i_1} *and* v_{i_2} *then* $|s_{i_1} - s_{i_2}|, |t_{i_1} - t_{i_2}| < 2\zeta$*. Then*

$$|\varrho([v_1]^\zeta_{s_1} \cap \ldots \cap [v_k]^\zeta_{s_k}) - \varrho([v_1]^\zeta_{t_1} \cap \ldots \cap [v_k]^\zeta_{t_k})| < \tau\lambda \tag{18}$$

if $|s_1 - t_1|, \ldots, |s_k - t_k| < \lambda$*.*

Proof. Assume (α), and let ϱ be induced by the τ-smooth $r\colon [-\frac{1}{2}, 1\frac{1}{2}]^N \to [0,1]$. Let $P, Q \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$ such that $\mathrm{d_H}(P,Q) = \lambda' < \lambda$. W.l.o.g. we assume $\varrho(\iota P) \geq \varrho(\iota Q)$. Then $|\varrho(\iota P) - \varrho(\iota Q)| = \inf_{p \in P} r(p) - \inf_{q \in Q} r(q)$. Let $\varepsilon > 0$, and choose an $q \in Q$ such that $|\varrho(\iota Q) - r(q)| \leq \varepsilon$; and choose $p \in P$ such that $\mathrm{d}_\infty(p,q) < \lambda' + \varepsilon$. Then $|\varrho(\iota P) - \varrho(\iota Q)| \leq r(p) - r(q) + \varepsilon < \tau(\lambda' + \varepsilon) + \varepsilon$. ($\beta$) follows.

Assume (β), and let $v_1, \ldots, v_k$ and $s_1, t_1, \ldots, s_N, t_N$ be as specified by condition (γ). By Lemma 4.10, we can write $[v_1]^\zeta_{s_1} \cap \ldots \cap [v_k]^\zeta_{s_k}$ as the disjunction of conjunctions of graded variables. Consider one of the disjuncts and assume that u_i, $1 \leq i \leq N$, occurs in it. Assume furthermore that u_i occurs at all places positively and that it occurs in v_j, $1 \leq j \leq k$. Then the disjunct can contain the conjunct $[u_i]_{s_j}$ or $[u_i]_{\geq s_j}$ or $[u_i]_{\leq s_j}$. If u_i occurs also in $v_{j'}$, $1 \leq j' \leq k$, then a further conjunct can be $[u_i]_{s_{j'}}$ or $[u_i]_{\geq s_{j'}}$ or $[u_i]_{\leq s_{j'}}$, and we have $|s_j - s_{j'}| < 2\zeta$. If u_i occurs at all places negatively instead, the same holds, but all values $s_j, s_{j'}, \ldots$ are to be replaced by ${\sim}s_j, {\sim}s_{j'}, \ldots$.

We conclude that the considered disjunct is of the form ιP for some non-empty $P \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$. We may furthermore decompose the expression $[v_1]^\zeta_{t_1} \cap \ldots \cap [v_k]^\zeta_{t_k}$ in exact analogy and consider the corresponding disjunct, which is of the form ιQ for some non-empty $Q \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$. By (17), $\mathrm{d_H}(P,Q) <$. So finally, (γ) follows by (16).

Assume (γ). Let us first show that (β) holds restricted to the case that P and Q are cubic and have a diameter $\leq 2\zeta$. So let $P, Q \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$ such that $\mathrm{d_H}(P,Q) < \lambda$. Then $P = I_1 \times \ldots \times I_N$ and $Q = J_1 \times \ldots \times J_N$ for some $I_1, \ldots, J_N \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}$ such that $\mathrm{d_H}(I_1, J_1), \ldots, \mathrm{d_H}(I_N, J_N) < \lambda$. We will show that there are $s_1, s_2, t_1, t_2 \in [-\zeta, 1+\zeta]$ such that $I_1 = U_\zeta(s_1) \cap U_\zeta(s_2)$ and $J_1 = U_\zeta(t_1) \cap U_\zeta(t_2)$ and $|s_1 - t_1|, |s_2 - t_2| < \lambda$. The same will follow for the remaining indices $2, \ldots, N$. Because, for $t \in [-\zeta, 1+\zeta]$, $\iota(U_\zeta(t) \times [-\frac{1}{2}, 1\frac{1}{2}] \times \ldots \times [-\frac{1}{2}, 1\frac{1}{2}]) = [u_1]_t$ and similarly for the indices $2, \ldots, N$, the assertion will follow.

If $I_1, J_1 \subseteq (0,1)$, the claim is easily seen to hold. If $[-\zeta, 0) \subseteq I_1, J_1$, then there are $s, t \in [-\zeta, \zeta)$ such that $I_1 = U_\zeta(s)$ and $J_1 = U_\zeta(t)$ and $|s - t| < \lambda$. Assume next that $[-\zeta, 0) \subseteq I_1$ but $[-\zeta, 0) \cap J_1 = \emptyset$. Let $I_1 = [-\zeta, r_1)$ and $J_1 = (r_2, r_3)$; then $\mathrm{d_H}(I_1, J_1) = r_2 \vee |r_3 - r_1|$. We have $I_1 = U_\zeta(r_1 - \zeta)$ and $J_1 = U_\zeta(r_3 - \zeta) \cap U_\zeta(r_2 + \zeta)$. As $r_1 - \zeta < \zeta$ and $r_2 < \lambda$, we may choose $r_2 < d < \lambda$ such that $r_1 - \zeta \leq r_2 + \zeta - d < \zeta$. Then $I_1 \subseteq U_\zeta(r_2 + \zeta - d)$. So we have $I_1 = U_\zeta(r_1 - \zeta) \cap U_\zeta(r_2 + \zeta - d)$, and the pairs $r_1 - \zeta, r_3 - \zeta$ and $r_2 + \zeta - d, r_2 + \zeta$ differ by a value $< \lambda$. The remaining cases are analogous.

We proceed to show (α). For $p \in [-\frac{1}{2}, 1\frac{1}{2}]^N$, let

$$r(p) \;=\; \sup\,\{\varrho(\iota P)\colon\; P \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N \text{ and } p \in P\}$$

We claim that r is τ-smooth. Indeed, let $p, q \in [-\frac{1}{2}, 1\frac{1}{2}]^N$ such that $\mathrm{d}_\infty(p,q) < \lambda$. Let λ' be such that $\mathrm{d}_\infty(p,q) < \lambda' < \lambda$ and let $\varepsilon > 0$. Choose cubic neighborhoods U_p of p and U_q of q with diameter $\leq 2\zeta$ such that $|r(p) - \varrho(\iota U_p)|, |r(q) - \varrho(\iota U_q)| \leq \varepsilon$ and $\mathrm{d}_\mathrm{H}(U_p, U_q) < \lambda'$. Then $|r(p) - r(q)| \leq |\varrho(\iota U_p) - \varrho(\iota U_q)| + 2\varepsilon < \tau\lambda' + 2\varepsilon$, and the claim follows.

We next claim that for $P \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$ we have $\varrho(\iota P) = \inf\,\{r(p) : \; p \in P\}$. Indeed, $\varrho(\iota P) \leq r(p)$ for any $p \in P$. Let $P \supset P_1 \supset P_2 \supset \ldots$ such that $\varrho(\iota P_i) = \varrho(\iota P)$ for all i and such that the diameter of P_i converges to 0. Let $\varepsilon > 0$; let i be large enough such that the diameter of P_i is below ε; then $\varrho(\iota Q) < \varrho(\iota P_i) + \tau\varepsilon = \varrho(\iota P) + \tau\varepsilon$ for any $Q \subseteq P_i$, and it follows $r(p) \leq \varrho(\iota P) + \tau\varepsilon$ for any $p \in P_i$. The claim follows, and (α) is shown. ☐

Lemma 5.4. *Let (M, ϱ) be a regular Kleene uncertainty algebra such that M is generated by $u_1, \ldots, u_N$; and let $\iota\colon\; \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N \to \mathcal{R}_M$ be defined according to Lemma* 5.1. *Let $R_1, \ldots, R_m$ be a partition of $\mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$; then $D = R_1 \cup \ldots \cup R_m$ is dense in $[-\frac{1}{2}, 1\frac{1}{2}]^N$. Let furthermore $r_1, \ldots, r_m \in [0,1]$ such that $r_i = 0$ for at least one index i. Let*

$$r\colon\; D \to [0,1], \quad p \mapsto \begin{cases} r_1 & \text{if } p \in R_1, \\ \ldots & \\ r_m & \text{if } p \in R_m, \end{cases}$$

and let ϱ be the rejection function induced by r. Furthermore, let

$$r'\colon\; [-\tfrac{1}{2}, 1\tfrac{1}{2}]^N \to [0,1], \quad p \mapsto \sup_{q \in D}\, (r(q) - \tau \mathrm{d}_\infty(p,q)) \vee 0,$$

and let ϱ' be the rejection function induced by r'. Then ϱ' is the smallest τ-smooth rejection function such that $\varrho' \geq \varrho$.

Moreover, let $P \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$. Then there are cubes $P_1, \ldots, P_n \in \mathcal{R}_{[-\frac{1}{2},1\frac{1}{2}]}{}^N$ such that $P = P_1 \vee \ldots \vee P_n$, and for each $i = 1, \ldots, n$ there is a cubic Q_i contained in R_j for some $j \in \{1, ..., m\}$ such that either

$$\varrho'(\iota P_i) \;=\; \varrho(\iota Q_i) - \tau \mathrm{d}_\mathrm{H}(P_i, Q_i) = \varrho'(\iota P) \tag{19}$$

or

$$\varrho'(\iota P_i) \geq \varrho(\iota Q_i) - \tau \mathrm{d}_\mathrm{H}(P_i, Q_i) > \varrho'(\iota P), \tag{20}$$

where the first case applies for at least one i.

Proof. We first show that r' is τ-smooth. Let $p, q \in [-\frac{1}{2}, 1\frac{1}{2}]^N$ such that $\mathrm{d}_\infty(p,q) < \lambda$. Let $\varepsilon > 0$, and choose $s_q \in D$ such that $r'(q) \leq r(s_q) - \tau\mathrm{d}_\infty(q, s_q) + \varepsilon$. Then $r'(p) \geq r(s_q) - \tau\mathrm{d}_\infty(p, s_q) \geq r(s_q) - \tau\mathrm{d}_\infty(p,q) - \tau\mathrm{d}_\infty(q, s_q) \geq r'(q) - \tau\mathrm{d}_\infty(p,q) - \varepsilon$. So $r'(q) - r'(p) < \tau\lambda$, and by symmetry we conclude $|r'(p) - r'(q)| < \tau\lambda$.

Clearly $r'|_D \geq r$, hence $\varrho' \geq \varrho$. Let now $\varrho'' \geq \varrho$ another τ-smooth rejection function. Let ϱ'' be induced by r''. Let $p \in D$, and let i be such that $p \in R_i$. Then $r''(p) \geq \varrho''(\iota R_i) \geq \varrho(\iota R_i) = r_i = r(p)$; hence $r'' \geq r$. For any $q \in D$, it follows $r''(p) \geq r''(q) - \tau\mathrm{d}_\infty(p,q) \geq r(q) - \tau\mathrm{d}_\infty(p,q)$; hence even $r'' \geq r'$.

It remains to show the last assertion. W.l.o.g. we may assume that $R_1, \ldots, R_m$ and P are all cubic. We have

$$\begin{aligned} \varrho'(\iota P) &= \inf_{p \in P} \sup_{q \in D} \left(r(q) - \tau\mathrm{d}_\infty(p,q)\right) \\ &= \inf_{p \in P} \max_i \left(r_i - \tau\mathrm{q}_\mathrm{H}(\{p\}, R_i)\right). \end{aligned} \tag{21}$$

Let us consider a point $p = (z_1, \ldots, z_N) \in P^-$. There are two cases:

(A) $\max_i\, (r_i - \tau\mathrm{q}_\mathrm{H}(\{p\}, R_i) = \varrho'(\iota P)$

(B) $\max_i\, (r_i - \tau\mathrm{q}_\mathrm{H}(\{p\}, R_i)) > \varrho'(\iota P)$.

If (A) applies, we will associate with p a cubic neighborhood U_p and a partition of $U_p \cap P$ such that, for each element U of this partition, $\varrho(\iota U)$ can be calculated according to (19). Note that by the continuity of the mapping $p \mapsto \max_i\, (r_i - \tau\mathrm{q}_\mathrm{H}(\{p\}, R_i)$ there is at least one $p \in P^-$ fulfilling (A). If (B) applies, we will associate with p a cubic neighborhood U_p such that (20) holds for $U_p \cap P$. $(U_p)_{p \in P^-}$ will be a cover of P^- by open sets; as P^- is compact, we may choose a finite subcover, and we will be done.

Case (A)*:* Let $J = \{j \in \{1, ..., m\} :\ \varrho'(\iota P) = r_j - \tau\mathrm{q}_\mathrm{H}(\{p\}, R_j)\}$. For $1 \leq i \leq N$, let $\pm_i \in \{+, -\}$ such that $E(\pm_1, ..., \pm_N) = \{(z_1 \pm_1 t_1, \ldots, z_N \pm_N t_N) : t_1, \ldots, t_N \geq 0\}$ intersects P non-emptily. Then there must be a $j \in J$ and an $\varepsilon > 0$ such that $U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N) \subseteq P$ and $\mathrm{q}_\mathrm{H}(\{(z_1 \pm_1 t, ..., z_N \pm_N t)\}, R_j) \leq \mathrm{q}_\mathrm{H}(\{p\}, R_j)$ for $0 < t \leq \varepsilon$; indeed, otherwise the infimum (21) would not be attained at p.

It follows that $\mathrm{q}_\mathrm{H}(\{q\}, R_j) \leq \mathrm{q}_\mathrm{H}(\{p\}, R_j)$ for all $q \in U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N)$, so that $\mathrm{q}_\mathrm{H}(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N), R_j) = \mathrm{q}_\mathrm{H}(\{p\}, R_j)$. We select a cubic $Q \subseteq R_j$ such that $\mathrm{q}_\mathrm{H}(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N), R_j) = \mathrm{d}_\mathrm{H}(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N), Q)$. So we have $\varrho'(\iota(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N)) = \varrho'(\iota P) = \varrho(\iota Q) - \tau\mathrm{d}_\mathrm{H}(U_\varepsilon(p) \cap E(\pm_1, ..., \pm_N), Q)$. We let $U_p = U_\varepsilon(p)$.

Case (B)*:* Let j be such that $r_j - \tau \mathrm{q_H}(\{p\}, R_j) > \varrho'(\iota P)$. Let U_p be a cubic neighborhood of p such that, for some r, we have $r_j - \tau \mathrm{q_H}(\{q\}, R_j) \geq r > \varrho'(\iota P)$ for all $q \in U_p$ and consequently $r_j - \tau \mathrm{q_H}(\{U_p \cap P\}, R_j) \geq R$. We select a cubic $Q \subseteq R_j$ such that $\mathrm{q_H}(\{U_p \cap P\}, R_j) = \mathrm{d_H}(\{U_p \cap P\}, Q)$. Then $\varrho'(\iota(U_p \cap P)) \geq \varrho(\iota Q) - \tau \mathrm{d_H}(\{U_p \cap P\}, Q) = r_j - \tau \mathrm{q_H}(\{U_p \cap P\}, R_j) > \varrho'(\iota P)$. □

We now modify $\mathtt{IG}^\zeta$ accordingly. The resulting logic will be called the Smooth Possibilistic Logic with Soft Gradation, denoted by $\mathtt{IG}^\zeta_\tau$.

Definition 5.5. The *propositions*, the set of which will still be denoted by $\mathcal{P}$, as well as the *implications* of $\mathtt{IG}^\zeta_\tau$ coincide with those of $\mathtt{IG}^\zeta$, respectively (see Def. 4.7).

An *evaluation* (v_f, v_b) of $\mathtt{IG}^\zeta_\tau$ in some regular Kleene uncertainty algebra (M, ϱ) is defined like for $\mathtt{IG}^\zeta$ except thatϱ is required to be τ-smooth w.r.t. $v_f(\varphi_1), \ldots, v_f(\varphi_N)$.

The notions of *satisfaction*, of a *theory*, and of *semantic entailment* for $\mathtt{IG}^\zeta$ is defined mutatis mutandis like for $\mathtt{I}$ (see Def. 2.3).

For an axiomatisation of $\mathtt{IG}^\zeta_\tau$ we have to add a rule reflecting the restriction to smooth rejection functions.

Definition 5.6. The rules of $\mathtt{IG}^\zeta_\tau$ are those of $\mathtt{IG}^\zeta$ (see Def. 4.8) and in addition the following *smoothing rule*, where $\psi_1, \ldots, \psi_k$ are gradable propositions such that each variable occurs in them at all places positively or at all places negatively; $s_1, \ldots, s_k, t_1, \ldots, t_k \in [-\zeta, 1+\zeta]$ such that $|s_1 - t_1|, \ldots, |s_k - t_k| < \lambda$ and if some variable occurs both in ψ_{i_1} and ψ_{i_2} then $s_{i_1} - s_{i_2}, t_{i_1} - t_{i_2} < 2\zeta$; α is graded proposition which has no variable in common with $\psi_1, \ldots, \psi_k$; and $d \in [0, 1]$:

$$\frac{(\psi_1, t_1), \ldots, (\psi_k, t_k) \overset{d}{\Rightarrow} \alpha}{(\psi_1, s_1), \ldots, (\psi_k, s_k) \overset{(d - \tau\lambda) \vee 0}{\Rightarrow} \alpha}$$

The notion of a proof as well as consistency is defined like for $\mathtt{IG}^\zeta$.

Theorem 5.7. *Let $\mathcal{T}$ be a consistent theory of $\mathtt{IG}^\zeta_\tau$ and $\Gamma \overset{e}{\Rightarrow} \delta$ an implication of $\mathtt{IG}^\zeta_\tau$. Then $\mathcal{T}$ semantically entails $\Gamma \overset{e}{\Rightarrow} \delta$ if and only if $\mathcal{T}$ proves $\Gamma \overset{e}{\Rightarrow} \delta$.*

Proof. The soundness of the rules of $\mathtt{IG}^\zeta_\tau$ follows from Theorem 4.14; the soundness of the smoothing rule follows from Lemma 5.3.

To show completeness, assume that $\mathcal{T}$ does not prove $\Gamma \overset{e}{\Rightarrow} \delta$. Disregarding the smoothness rule, we proceed like in the proof of Theorem 4.14 to construct the evaluation (v_f, v_b) in the regular Kleene uncertainty algebra (M, ϱ) such that all elements of $\mathcal{T}$ are satisfied by (v_f, v_b). Let furthermore ϱ' be the smallest τ-smooth rejection function such that $\varrho' \geq q\varrho$; then $\mathcal{T}$ is satisfied by (v_f, v_b) also in (M, ϱ'). Moreover, let α be any proposition and $d = \varrho'(v_b(\alpha))$. By Lemma 5.4 and the presence of the smoothness rule, $\mathcal{T}$ proves $\alpha \stackrel{d}{\Rightarrow} \bot$.

It follows that if α is the conjunction of $\Gamma \cup \{\neg\delta\}$, then $d < e$. Hence $\Gamma \stackrel{e}{\Rightarrow} \delta$ is not satisfied in (M, ϱ'). □

6 Conclusion

We have extended Dubois and Prade's Possibilistic Logic so as to allow the treatment of vague notions. Our guideline was to integrate, but not to mix, aspects of uncertainty and of vagueness in a uniform framework. Statements of the form that a property holds to a specific degree were integrated into the plausibility-based calculus. The degree of presence of a property has by default no influence on the degree of its plausibility; a smoothness rule, whose effect can be controlled by a real parameter, can however be added to ensure the continuity of the degree of uncertainty with regard to changes of the degrees of presence of the involved properties.

We note that this way of treating vagueness can, as we suppose, be applied to any logic other than Possibilistic Logic as well. From the foundational point of view, the method has, as we guess, the advantage that fuzzy sets are treated as parametrised sets of crisp properties, which in turn are treated classically. The question how to model vague properties by fuzzy sets is however assumed to be solved and the Kleene algebra structure has to be accepted as definitional. We may just underline that the choice of an appropriate fuzzy set for a given property works in practice very well and the decision about the shape of fuzzy sets can in fact be put on firm grounds as, for instance, the work [HeCa] demonstrates. Even the adequacy of the Kleene algebra structure is supported by results of [HeCa]. But we should certainly remain cautious – in general we should say that fuzzy sets endowed with the standard operations are widely used but purely pragmatically.

The remainder of the paper is devoted to a practical application. Our formalism can serve to endow the medical expert system CADIAG-2, which we have already mentioned in the introductory chapter, with a clear theoretical basis. The formalism which results might not exactly coincide with the original system, but reflects its concepts quite well. The details will be elaborated elsewhere; here we will only roughly outline the idea.

For a general description of CADIAG-2 see, for instance, [AdKo]; for formally oriented descriptions, see [CiVe, Pic]. We shall demonstrate by an example how the inference mechanism of CADIAG-2 can be mimicked in our logic $\mathtt{IG}^{\zeta}_{\tau}$. We assume that the t-norm used by CADIAG-2 is the Łukasiewicz t-norm, and we put $\tau = 1$.

Assume that the following facts about a patient are known and that the following rule is contained in the knowledge base of CADIAG-2; we use the notation of [CiVe]:

$$(\sigma_1,\ s), \quad (\sigma_2,\ t), \quad (\sigma_1 \wedge \sigma_2 \to \delta,\ d);$$

here, σ_1 and σ_2 denote symptoms, δ denotes a disease, and $s, t, d \in [0, 1]$. These statements code the following information: the symptom σ_1 holds to the degree s; the symptom σ_2 holds to the degree t; and if the conjunction of these two symptom evaluate to 1, that is, if they both fully apply, we may conclude that δ is certain to the degree d. The following rules of the logic underlying CADIAG-2 – here we show the appropriate instances – are applied to draw a conclusion in case that $s, t, d > 0$ (see [CiVe]):

$$\frac{(\sigma_1,\ s) \quad (\sigma_2,\ t)}{(\sigma_1 \wedge \sigma_2,\ s{\wedge}t)} \qquad \frac{(\sigma_1 \wedge \sigma_2,\ s{\wedge}t) \quad (\sigma_1 \wedge \sigma_2 \to \delta,\ d)}{(\delta,\ d \star (s{\wedge}t))},$$

where $\star$ is a t-norm. By default, the Gödel t-norm is used, but here we assume that the Łukasiewicz t-norm is used:

$$\star\colon\ [0, 1] \times [0, 1] \to [0, 1], \quad (s, t) \mapsto (s + t - 1){\vee}0.$$

In the framework proposed in the present paper, we may formulate the rule contained in the knowledge base as

$$(\sigma_1 \wedge \sigma_2, 1) \stackrel{d}{\Rightarrow} (\delta, 1);$$

from this implication of $\mathtt{IG}^{\zeta}_{\tau}$ we derive

$$(\sigma_1, 1), (\sigma_2, 1) \stackrel{d}{\Rightarrow} (\delta, 1),$$

and using the smoothing rule furthermore

$$(\sigma_1, s), (\sigma_2, t) \stackrel{(d - \tau(1 - s \wedge t)) \vee 0}{\Rightarrow} (\delta, 1),$$

which, when putting $\tau = 1$, describes the same conclusion as shown above for CADIAG-2.

The progress of this approach compared to [CiVe] as well as to [Pic] is that it both uncertainty and gradedness are appropriately taken into account. CADIAG-2 does not clearly distinguish between degrees of presence and degrees of certainty; this issue is solved in the present approach.

Like in [CiVe] and [Pic] however the problem appears that we offer a sound semantics but a strictly stronger logic. The additional strength might be justifiable though. All in all, our example indicates that the way inferences are realised in CADIAG-2 are well compatible with the calculus $\mathtt{IG}_\tau^\zeta$ presented in this paper.

Acknowledgments

The author acknowledges partial support of the Eurocores-LogICCC ESF project LoMoReVI.

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